STRONG CARLEMAN OPERATORS ARE OF HILBERT-SCHMIDT TYPE

JOACHIM WEIDMANN

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This is the solution of a problem posed by G. Targonski [6, §13 V]: Do unbounded strong Carleman operators exist? In fact, we shall prove that every strong Carleman operator is a Hilbert-Schmidt operator (hence it is certainly bounded).

1. Definitions and known results. Carleman operators are usually defined in the space $L_2(a, b)$ where $a \leq -\infty, b \leq \infty$; without any restriction of generality we may assume that $-\infty < a < b < \infty$. There are several definitions of a Carleman operator used in the literature (e.g. T. Carleman [1], M. Stone [5], G. Targonski [6]; for “semi-Carleman operators” see M. Schreiber [4]). We shall mainly follow the definition used by G. Targonski, but in addition to his definition we shall assume that a Carleman operator is densely defined (e.g. §3).

DEFINITIONS. A densely defined operator $K$ in the Hilbert space $L_2(a, b)$ is called a Carleman operator if it allows a representation of the form

$$(Kf)(x) = \int_a^b K(x, y)f(y)dy$$

for almost all $x$, where $\int_a^b |K(x, y)|^2dy < \infty$ for almost all $x$. The domain of $K$ consists of all elements $f \in L_2(a, b)$ such that $\int_a^b K(x, y)f(y)dy$ (which is defined for almost all $x$) represents an element of $L_2(a, b)$.

An operator $K$ is called a strong Carleman operator if $UKU^*$ is a Carleman operator for every unitary operator $U$. An operator $K$ in a Hilbert space is a Hilbert-Schmidt operator (or $K$ is of Hilbert-Schmidt type) if for every orthonormal system $(\phi_n)$, $\sum_n |K\phi_n|^2 < \infty$ (this series has the same value for all complete orthonormal systems).

It is known (e.g. [6]) that every Hilbert-Schmidt operator is a strong Carleman operator. In [6] it is also shown that bounded strong Carleman operators are of Hilbert-Schmidt type. Using the result of this note we may say: An operator in $L_2(a, b)$ is a strong Carleman operator if and only if it is a Hilbert-Schmidt operator.

We shall use the following known results:

THEOREM I ([6, Lemmata 9.1 and 9.2]). If $K$ is a strong Carleman operator and $B$ is bounded, then $BK$ and $KB$ are strong Carleman operators.
THEOREM II ([3, SATZ 4], [6, PROOF OF LEMMA 9.5]). For every selfadjoint Carleman operator, 0 is a limit point of its spectrum; the spectrum of a selfadjoint strong Carleman operator has at most the limit points $-\infty$, 0 and $\infty$.

THEOREM III ([2, VI.2.7]). A densely defined closed operator $K$ in a Hilbert space can be factorized as $K = UT$, where $T$ is selfadjoint (non-negative) and $U$ is a partial isometry with initial set $\text{Cl}(\mathcal{R}(T))$ and final set $\text{Cl}(\mathcal{R}(K))$ ($\text{Cl} =$ closure).

2. Proofs. The proof of our first theorem is almost the same as the proof of [6, Theorem 9.2].

THEOREM 1. Every selfadjoint strong Carleman operator is of Hilbert-Schmidt type.

PROOF. Let $K$ be a selfadjoint strong Carleman operator. Theorem II asserts that the spectrum of $K$ consists of a sequence $(\lambda_n)$ of eigenvalues with limit point 0 (and eventually $\pm \infty$). Since $K$ is a strong Carleman operator there exists for any complete orthonormal system $(\phi_n)$ a unitary transformation $U$ and a kernel $K_U(x, y)$ such that

$$\int_a^b |K_U(x, y)|^2 dy < \infty,$$

$$(UKU^* \rho_n)(x) = \lambda_n \rho_n(x), \quad (UKU^* f)(x) = \int_a^b K_U(x, y)f(y)dy$$

for almost all $x$ and $f \in \mathcal{D}(UKU^*) = \mathcal{D}(K)$. This implies that $\lambda_n \rho_n(x)$ are the Fourier coefficients of the $L_2$-function $K_U(x, y)$ (as a function of $y$) with respect to the complete orthonormal system $(\rho_n)$. Since $K_U(x, y)$ (as a function of $y$) is in $L_2(a, b)$ for almost all $x$, this implies $\sum_n |\lambda_n \rho_n(x)|^2 < \infty$ for almost all $x$. Let us now chose the complete orthonormal system

$$\rho_n(x) = (b - a)^{-1/2} \exp\{2\pi i nx/(b - a)\};$$

it follows that $\sum_n |\lambda_n|^2 < \infty$, i.e. $K$ is a Hilbert-Schmidt operator.

THEOREM 2. Every Carleman operator is closed.

PROOF. Let $K$ be a Carleman operator, $(u_n) \subset \mathcal{D}(K)$, $u_n \to u$, $Ku_n \to v$. We may write

$$(Kw)(x) = F[z][w] \quad \text{for almost all } x, w \in \mathcal{D}(K),$$

where $F[z]$ is a family of bounded linear functionals in $L_2(a, b)$. Obviously
(Ku_n)(x) - F_x[u] = F_x[u_n] - F_x[u] \to 0 \quad \text{for almost all } x.

By assumption Ku_n \to v in L_2(a, b); hence there exists a subsequence (u_{n_k}) of (u_n) such that (Ku_{n_k})(x) - v(x) \to 0 for almost all x. This implies that v(x) = F_x[u] for almost all x, i.e. u \in \mathcal{D}(K) and Ku = v.

**Theorem 3.** Every strong Carleman operator is a Hilbert-Schmidt operator.

**Proof.** Let K be a strong Carleman operator; then K is closed by Theorem 2. Hence by Theorem III K = UT where T is selfadjoint and U is a partial isometry with initial set Cl(R(T)) and final set Cl(R(K)). Then U^*U is a partial isometry with initial and final set Cl(R(T)), hence T = U^*K. By Theorem I T is a selfadjoint strong Carleman operator and consequently by Theorem 1 T is of Hilbert-Schmidt type. Hence K = UT is a Hilbert-Schmidt operator.

3. Remarks on operators which are not densely defined. It is possible to neglect "densely defined" in the definition of a Carleman operator. The kernel \( K(x, y) = g(x)h(y) \) (\( g \in L_2(a, b), h \in L_2(a, b) \)) for example defines a Carleman operator of this type:

\[ Kf = 0 \quad \text{if } (f, h) = 0 \]
\[ = \text{not defined} \quad \text{if } (f, h) \neq 0. \]

The main disadvantage of these operators is the fact that the kernel is not uniquely determined by the operator (in the above example, \( g \) is an arbitrary function not contained in \( L_2(a, b) \)).

Let K be a strong Carleman operator (in the corresponding sense, i.e. not necessarily densely defined) then \( KP \) is a strong Carleman operator, where \( P \) is the orthogonal projection onto \( Cl(D(K)) \). Since \( KP \) is densely defined we may apply the results of §2 and find: \( KP \) is a Hilbert-Schmidt operator.

**References**


**Institut für Angewandte Mathematik, University of Heidelberg**