PLANARITY IN ALGEBRAIC SYSTEMS

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Planarity was introduced into algebra by Marshall Hall in his well-known coordinatization of a projective plane by a planar ternary ring [4]. In [6], J. L. Zemmer defines a near-field to be planar when the equation $ax - bx + c$ has a unique solution for $a \neq b$. In our investigation of planarity, we discovered that if $(N, +, \cdot)$ is a near-ring satisfying the above equational property, then $(N, +, \cdot)$ is a near-field. (This was conjectured by both D. R. Hughes and J. L. Zemmer in private communications.) We present some extensions of this result together with geometric interpretations of “planar” near-rings.

Definitions and notations. By a left distributive system is meant a triple $(N, +, \cdot)$ such that multiplication $\cdot$ is left distributive over addition $+$. Elements $a, b \in N$ are called left equivalent multipliers, denoted by $a \equiv_m b$ if $ax = bx$ for all $x \in N$. The relation $\equiv_m$ is discrete when $a \equiv_m b$ implies $a = b$. A left distributive system is said to possess the planar property if the equation $ax - bx + c$ has a unique solution for $a \neq b$.

DEFINITION. A left distributive system $(N, +, \cdot)$ with planar property is a planar system if

1. in $(N, +)$ the right cancellation law is valid;
2. in $(N, +)$ there is an identity $0$;
3. $(N, \cdot)$ is a semi-group;
4. there are at least three points in $N$, no two of which are left equivalent multipliers.

A planar system is integral if $0$ is the only left zero divisor.

Main results. Let $(N, +, \cdot)$ be an integral planar system. Then $0 \cdot x = x \cdot 0 = 0$ for all $x \in N$. Let $1_a$ be the solution to the equation $a \cdot x = a$, $a \neq 0$, and $B_a = \{x \in N^* | x \cdot 1_a = x\}$, where $N^*$ denotes the nonzero elements of $N$. We have the following

THEOREM 1. Let $(N, +, \cdot)$ be an integral planar system. Then
(i) each $(B_a, \cdot)$ is a group with identity $1_a$;
(ii) the family $\{B_a\}_{a \in N^*}$ is pairwise disjoint;
(iii) $N^* = \bigcup_{a \in N^*} B_a$.

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(iv) \( N^*B_a = B_a \) for each \( a \in N^* \);
(v) if \( a, c \in N^* \), then \( \phi: B_a \rightarrow B_c \) defined by \( \phi(x) = x1_c \) is an isomorphism;
(vi) each \( 1_a \) is a left identity for \( (N, +, \cdot) \).

**Corollary.** Let \( (N, +, \cdot) \) be a near-ring that is an integral planar system with \( =_m \) discrete. Then \( (N, +, \cdot) \) is a planar near-field.

**Proof.** If \( a, b \in N^* \), then \( 1_a = 1_b \).

In the sequel a near-ring that is an integral planar system will be called an integral planar near-ring.

**Theorem 2.** Suppose \( (N, +, \cdot) \) is an integral planar near-ring and each \( B_a = \{0\} \cup B_a \) is an additive normal subgroup. Also suppose that no \( B_a = N \) but any two \( B_a, B_b \) generate \( N \) under \( + \). Then

(i) each \( (B_a, +, \cdot) \) is a near-field;
(ii) \( (B_a, +, \cdot) \) is isomorphic to \( (B_c, +, \cdot) \) if \( (x+y)1_e = x1_c + y1_e \) for all \( x, y \in B_a \);
(iii) \( (N, +) \) is abelian and is isomorphic to the direct sum \( B_a \oplus B_e \) as groups;
(iv) the points of \( N \) are the points of an affine plane \( A \) with the cosets of the \( B_a \) as lines;
(v) the plane \( A \) can be coordinatized by a skew field.

**Proof.** The group \( (N, +) \) is a \( \Phi(I, IV) \) group \([5]\). A \( \Phi(I, IV) \) group is abelian since \( x \rightarrow x + g \) induces a translation on \( A \) and so Axiom 4a is satisfied (p. 58 of \([1]\)). Axiom 4bP (p. 63 of \([1]\)) holds at \( 0 \in N \) where \( x \rightarrow tx \) are the required dilatations.

**Theorem 3.** Suppose \( (N, +, \cdot) \) is a finite integral planar near-ring and each \( B_a = \{0\} \cup B_a \) is an additive subgroup. Also suppose that no \( B_a = N \) but any two \( B_a, B_b \) generate \( N \) under \( + \). Then

(i) \( (N, +) \) is abelian;
(ii) the affine plane \( A \) determined by \( N \) can be coordinatized by a field \((F, +, \cdot)\);
(iii) each \( (B_a, +, \cdot) \) is a field;
(iv) each \( B_a = \{(x, mx) | x \in F\} \) for some \( m \in F \), or \( B_a = \{(0, x) | x \in F\} \).

**Proof.** Each \( (B_a, +, \cdot) \) is a near-field, hence \( (N, +) \) is a \( p \)-group. Now \( (B_a, +) \) is contained in the center of \( (N, +) \) for some \( a \in N^* \), hence \( (N, +) \) is abelian since \( N = B_a + B_e \). A finite skew field is a field, and each \( (B_a, +, \cdot) \) is isomorphic to the coordinization skew field.

**Examples.** 1. Let \( (F, +, \cdot) \) be a field. Define \( +_\lambda \) (\( \lambda \neq 0 \)) by \( a +_\lambda b = b \) if \( a = 0 \), \( a +_\lambda b = a + (\lambda b) \) when \( a \neq 0 \). Then \( (F, +_\lambda, \cdot) \) is a nontrivial
integral planar system where \( \otimes \) is discrete and \(+\) is not necessarily associative.

2. Let \((\mathbb{R} \times \mathbb{R}, +)\) be additive group of complex numbers. Define \(\cdot\) by 
   \[(a, b) \cdot (c, d) = \| (a, b) \| (c, d)\]
   where \(\| - \|\) is any norm on \(\mathbb{R} \times \mathbb{R}\). Then 
   \((\mathbb{R} \times \mathbb{R}, +, \cdot)\) is an integral planar near-ring.

3. Let \((\mathbb{R} \times \mathbb{R}, +)\) be as in 2. Define \(\cdot\) by 
   \[(a, b) \cdot (c, d) = (a, b)^\wedge (c, d)\]
   where 
   \[(a, b)^\wedge = 0 \text{ if } a = b = 0; \text{ otherwise } (a, b)^\wedge \text{ is the first nonzero coordinate.}\]
   Then \((\mathbb{R} \times \mathbb{R}, +, \cdot)\) is an integral planar near-ring.

4. Let \((\mathbb{R} \times \mathbb{R}, +)\) be as in 2. Define \(*\) by 
   \[(a, b) * (c, d) = (a, b)/\|(a, b)\| \cdot (c, d)\]
   where 
   \[(a, b) = (a^2 + b^2)^{1/2} \neq 0 \text{ and } \cdot \text{ denotes the usual multiplication of complex numbers.}\]
   If \((a, b) = (0, 0)\), then 
   \[(a, b)^* (c, d) = (0, 0)\]. Then \((\mathbb{R} \times \mathbb{R}, +, \cdot)\) is an integral planar near-ring.

5. Table 1 defines a multiplication \(\cdot\) on the cyclic group \((\mathbb{Z}_5, +)\) such that 
   \((\mathbb{Z}_5, +, \cdot)\) is an integral planar near-ring. Note that 
   \[B_1 = \{1, 4\}, B_2 = \{2, 3\}.\]
   Define 
   \[B_i = B_i \cup \{0\} \text{ and } B_{ij} = B_i + j, \ i = 1, 2; j \in \mathbb{Z}_5.\]

\[
\begin{array}{cccc}
  \cdot & 0 & 1 & 2 & 3 & 4 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 2 & 3 & 4 \\
 2 & 0 & 4 & 3 & 2 & 1 \\
 3 & 0 & 1 & 2 & 3 & 4 \\
 4 & 0 & 4 & 3 & 2 & 1 \\
\end{array}
\]

Table 1

we let \(I = \mathbb{Z}_5\), then the \(B_{ij}\) are circles of an inverse plane [3]. This example was obtained using a digital computer. (See [2].)

It is of interest to graph the left identities and the \(B_a\) in each of the Examples 2, 3, and 4.

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