NONEXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS OF NONLINEAR EIGENVALUE PROBLEMS

BY HERBERT B. KELLER

Communicated by E. Isaacson, March 18, 1968

We consider nonlinear eigenvalue problems of the general form:

(1) \[ Lu = F(\lambda, x, u), \quad x \in D, \]

(2) \[ \beta(x) \frac{\partial u}{\partial \nu} + \alpha(x) u = 0, \quad x \in \partial D. \]

Here \( x = (x_1, x_2, \ldots, x_m) \) and

\[
L\phi \equiv \sum_{i,j=1}^{m} \partial_{i}[a_{ij}(x)\partial_{j}\phi] - a_0(x)\phi, \quad a_{ij}(x) = a_{ji}(x) \quad x \in D;
\]

\[
\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq a^2 \sum_{i=1}^{m} \xi_i^2, \quad a^2 > 0; \quad a_0(x) \equiv 0
\]

(3) \[
\frac{\partial \phi}{\partial \nu} \equiv \sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\partial_{j}\phi \quad \alpha(x)\beta(x) \geq 0, \quad \alpha(x) \neq 0, \quad \alpha(x) + \beta(x) > 0 \quad x \in \partial D.
\]

All coefficients and the derivatives of the \( a_{ij}(x) \) are continuous on the appropriate closed sets \( \overline{D} \) or \( \partial D \), and the latter is piecewise smooth with exterior unit normal vector \( (n_1(x), n_2(x), \ldots, n_m(x)) \) at \( x \in \partial D \). We first prove a simple but useful result on conditions for the nonexistence of positive solutions of (1)–(2).

**Theorem 1.** Let \( F(\lambda, x, z) \) be continuous on \( x \in D, z > 0 \). For any positive continuous function \( r(x) \) on \( \overline{D} \), let \( \mu_1 \{ r \} \) be the least eigenvalue of

---

1 This work was supported under Contract DAHC 04-68-C-0006 with the U. S. Army Research Office (Durham).

887
\[ L\psi + \mu r(x)\psi = 0, \quad x \in D, \]
\[ \beta(x)\psi/\partial v + \alpha(x)\psi = 0, \quad x \in \partial D. \]

Then (1)–(2) has no positive solution for any \( \lambda \in \Lambda \{r\} \) where

\[ \Lambda \{r\} = \{ \lambda \mid F(\lambda, x, z) + \mu_1 \{r\} r(x)z \neq 0, \quad \text{all } x \in D, z > 0 \}. \]

**Proof.** Suppose (1)–(2) has a positive solution, \( u(x) > 0, x \in D \), for a given fixed \( \lambda \). Then this solution trivially satisfies

\[ Lu + \mu_1 \{r\} r(x)u = F(\lambda, x, u) + \mu_1 \{r\} r(x)u \]

and (2). Since \( L \) is selfadjoint, the right-hand side must be orthogonal to \( \psi_1(x) \), the eigenfunction of (4) belonging to \( \mu_1 \{r\} \). From (3) it follows that \( \psi_1(x) \) is of one sign on \( D \). Thus the orthogonality relation requires that the continuous right-hand side change sign on \( D \). Hence \( \lambda \in \Lambda \{r\} \).

Of course piecewise continuous \( F(\lambda, x, u) \) and \( r(x) > 0 \) are easily included by replacing \( \neq 0 \) in definition (5) by either alternative: \( > 0 \) or \( < 0 \). The above theorem generalizes some nonexistence results contained in Keller & Cohen [1].

We now consider some special cases of (1)–(2) in which positive solutions are known or conjectured to exist. The problems are of the form:

\[ Lu + \lambda r(x)u = f(x, u), \quad x \in D, \]
\[ \beta(x)\partial u/\partial v + \alpha(x)u = 0, \quad x \in \partial D, \]

where \( r(x) \) is continuous and positive on \( D \).

Some nonexistence results for the above problem are a simple consequence of Theorem 1.

**Corollary 1.1.** (a) For some constant \( k \) let \( f(x, z) > kr(x)z \) for all \( z > 0 \) and \( x \in D \). Then (6) has no positive solutions for any \( \lambda \leq \mu_1 \{r\} + k \).

(b) For some constant \( k \) let \( f(x, z) < kr(x)z \) for all \( z > 0 \), and \( x \in D \). Then (6) has no positive solution for any \( \lambda \leq \mu_1 \{r\} + k \).

**Proof.** (a) \( F(\lambda, x, z) \equiv f(x, z) - \lambda r(x)z > (k - \lambda) r(x)z \geq -\mu_1 \{r\} r(x)z \) for \( z > 0 \) if \( \lambda \leq \mu_1 \{r\} + k \). Then \( \lambda \in \Lambda \{r\} \).

(b) As above, we see that \( F(\lambda, x, z) < -\mu_1 \{r\} r(x)z \) if \( \lambda \geq \mu_1 \{r\} + k \).

Note that \( k \) in the Corollary may have either sign, but the case \( k = 0 \) is of particular interest. It implies that if (6) is to have positive solutions for all \( \lambda \geq 0 \), then \( f(x, z) \) must change sign on \( z > 0, x \in D \).

In a recent paper D. S. Cohen [2] proves that (6) has unique positive solutions for \( 0 \leq \lambda < \mu_1 \{r\} \) when \( f(x, z) \equiv -f(x) + g(x, z) \) where:
\[ f(x) < 0, \quad g(x, z) > 0, \quad g^*(x, z) > 0, \quad g^*_*(x, z) > 0 \quad \text{and} \quad g^*_*(x, z) > g(x, z) \text{ for} \]
all \( z > 0, \, x \in D \). It can be shown that if \( f(x, 0) < 0, \, f_s(x, z) > 0 \) for all \( z > 0, \, x \in D \) and \( \lim_{z \to \infty} f_s(x, z) = \infty \), then (6) has positive solutions for all \( \lambda \). Under these conditions D. Cohen has observed that a result of Levinson [3] implies that (6), with \( L \equiv \Delta \) and \( \beta \equiv 0 \), has solutions for all values of \( \lambda \). We shall show that positive solutions of (6) are unique if only \( f_s(x, z) \) is increasing in \( z \) for \( z > 0 \) and \( f(x, 0) \leq 0 \).

**Theorem 2.** Let \( f(x, z) \) have a continuous \( z \)-derivative and satisfy for all \( x \in D \):

\[
\begin{align*}
\text{(a) } & f(x, 0) = f_0(x) \leq 0, \\
\text{(b) } & f_s(x, z) > f_s(x, z') > 0 \quad \text{if } z > z' > 0.
\end{align*}
\]

Then positive solutions of (6) are unique (for all \( \lambda \) for which they exist).

**Proof.** Assume \( u(x) \) and \( v(x) \) are distinct positive solutions of (6) for the same value of \( \lambda \). Then since \( f_s(x, z) \) is continuous for \( z > 0 \), we have

\[
 f(x, u(x)) - f(x, v(x)) = q(x, u(x), v(x)) [u(x) - v(x)]
\]

where

\[
 q(x; u, v) = \int_0^1 f_s(x, tu(x) + (1 - t)v(x)) \, dt.
\]

Thus with \( w(x) = u(x) - v(x) \), we obtain from (6) for \( u \) and \( v \):

\[
\begin{align*}
Lw + [\lambda r(x) - q(x; u, v)]w &= 0, \quad x \in D, \\
\beta(x) \partial w / \partial v + \alpha(x) w &= 0, \quad x \in \partial D.
\end{align*}
\]

Noting that \( f(x, u(x)) - f(x, 0) = q(x; u(x), 0)u(x) \) we can write (6) as

\[
\begin{align*}
L u + [\lambda r(x) - q(x; u, 0)]u &= f_0(x), \quad x \in D, \\
\beta(x) \partial u / \partial v + \alpha(x) u &= 0, \quad x \in \partial D.
\end{align*}
\]

Now consider the two eigenvalue problems, with eigenvalue parameters \( \sigma \) and \( \tau \):

\[
\begin{align*}
L \phi + [\sigma r(x) - q(x; u, v)] \phi &= 0, \quad x \in D, \\
\beta(x) \partial \phi / \partial \nu + \alpha(x) \phi &= 0, \quad x \in \partial D; \\
L \psi + [\tau r(x) - q(x; u, 0)] \psi &= 0, \quad x \in D, \\
\beta(x) \partial \psi / \partial \nu + \alpha(x) \psi &= 0, \quad x \in \partial D.
\end{align*}
\]

The least eigenvalue, \( \sigma_1 \) and \( \tau_1 \) respectively, of each of these problems can be characterized by the variational principle:
\[ \sigma_1 = \min_{\phi \in \mathcal{A}} \left\{ Q[\phi] + \int_D \int q(x; u, v) \phi^2(x) \, dx \right\} / H[\phi], \]

\[ \tau_1 = \min_{\phi \in \mathcal{A}} \left\{ Q[\phi] + \int_D \int q(x; u, 0) \phi^2(x) \, dx \right\} / H[\phi]. \]

Here the class of admissible functions is, say, \( \mathcal{A} = \{ \phi \in C(D) \cap C'(\Omega) : \phi(x) = 0, x \in \partial D_1 \} \) where \( \beta(x) = 0 \) if and only if \( x \in \partial D_1, \partial D = \partial D_1 \cup \partial D_2, \partial D_1 \cap \partial D_2 = \emptyset \) and:

\[ Q[\phi] = \int_D \int \left( \sum_{i,j=1}^m a_{ij}(x) \partial_i \phi \partial_j \phi + a_0(x) \phi^2 \right) \, dx + \int_{\partial D} \frac{\alpha(x)}{\beta(x)} \phi^2 \, ds, \]

\[ H[\phi] = \int_D \int r(x) \phi^2 \, dx. \]

Since \( f_s(x, z) \) is increasing in \( z \) for \( s > 0 \) and \( v(x) > 0 \) on \( D \), we must have for all \( x \in D, \)

\[ q(x; u(x), v(x)) > q(x; u(x), 0). \]

Thus from the above variational principle it follows that

\[ \sigma_1 > \tau_1. \]

By assumption, \( w(x) \neq 0 \), and so the parameter \( \lambda \) appearing in (9) must be some eigenvalue of the problem (11a). Since \( \sigma_1 \) is the least eigenvalue of that problem we must have \( \lambda \geq \sigma_1 > \tau_1 \). Now write (10) as:

\[ Lu + [\tau_1 r(x) - q(x; u, 0)] u = f_0(x) + [\tau_1 - \lambda] r(x) u(x), \quad x \in D, \]

\[ \beta(x) \partial u / \partial v + \alpha(x) u = 0, \quad x \in \partial D. \]

But \( \tau_1 \) is the least eigenvalue of (11b) and so the right-hand side in the above differential equation must be orthogonal to \( \psi_1(x) \), the eigenfunction belonging to \( \tau_1 \). However, this is impossible since \( \psi_1(x) \) is of one sign on \( D \) and, since \( u(x) \) is a positive solution,

\[ f_0(x) + (\tau_1 - \lambda) r(x) u(x) < 0 \quad \text{on } D. \]

The contradiction implies \( w(x) = 0 \). \( \square \)

The above proof remains valid if we relax the monotonicity condition (7b) to just nondecreasing derivative, \( f_s(x, z) \geq f_s(x, z'), z > z' > 0; \) but strengthen condition (7a) to \( f_0(x) < 0 \). Clearly our result also applies to the case with \( f \equiv f(\lambda, x, u) \) provided (7) holds for the appropriate values of \( \lambda \).
Many additional results have been obtained under the hypothesis of Theorem 2; namely: (i) positive solutions of (6) are increasing functions of \( \lambda \) for all \( x \in D \); (ii) the set of \( \lambda \) for which positive solutions of (6) exist is open above; (iii) if \( f_0(x) = 0 \), then (6) has no positive solutions for all \( \lambda \leq \lambda_1 \) where \( \lambda_1 \) is the least eigenvalue of

\[
L\phi + [\lambda r(x) - f_4(x; 0)]\phi = 0, \quad x \in D,
\]

\[
\beta(x)\frac{\partial \phi}{\partial n} + \alpha(x)\phi = 0, \quad x \in \partial D;
\]

(iv) if \( f_0(x) < 0 \) on \( D \), then (6) has positive solutions for all \( \lambda < \lambda_1 \); (v) if \( f_0(x) < 0 \) on \( D \) and a positive solution of (6) exists for some \( \lambda' \), then positive solutions exist for all \( \lambda \leq \lambda' \).

Also, we can show that (6) has a positive solution for arbitrarily large \( \lambda \) if in addition to (7) and \( f_0(x) < 0 \) on \( D \) we have \( \lim_{x \to \infty} f_4(x, z) = +\infty \) on \( D \). Combined with (v) above and Theorem 2 this yields unique positive solutions of (6) for all \( \lambda \). The results in (i)–(v) are proven by combining the technique in Theorem 2 with the use of the Positivity Lemma as in [1], and are thus constructive results. Variational procedures are employed to show existence for arbitrarily large \( \lambda \). The detailed proofs will be given elsewhere.

**REFERENCES**

