1. Let $L^p$ denote the Lebesgue space for the normalized measure $(1/2\pi)d\theta$ defined on the unit circle $T = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, let $H^p$ denote the corresponding Hardy space of functions in $L^p$ which have zero negative Fourier coefficients and let $P$ be the projection of $L^2$ onto $H^2$. For $\phi$ in $L^\infty$ we define the Toeplitz operator $T_\phi$ on $H^2$ by $T_\phi f = P(\phi f)$ for $f$ in $H^2$. It is clear that $T_\phi$ is bounded and this class of operators has been much studied.

Much of the interest in Toeplitz operators has been directed to the determination of their spectra. For $\phi$ in $L^\infty$ it was shown by Wintner in [12] that the spectrum $\sigma(T_\phi)$ of $T_\phi$ is the closure of the range of the analytic extension of $\phi$ to the unit disk $D$. For $\phi$ in the space $C$ of continuous functions on $T$ it was shown by Devinatz [4] (see [10] for earlier results) that $\sigma(T_\phi)$ consists of the range of $\phi$ on $T$ along with those $\lambda$ for which the index of $\lambda$ with respect to the curve determined by $\phi$ is different from zero. In this note we describe $\sigma(T_\phi)$ for $\phi$ in the linear span of $L^\infty$ and $C$. (This manifold is actually a closed subalgebra of $L^\infty$.) We show that such a $T_\phi$ is invertible if its harmonic extension $\phi$ to $D$ is bounded away from zero on a neighborhood of $T$ and the index of the curve $\phi(Re^{i\theta})$ is zero for $R$ sufficiently large. Our technique can be viewed as an extension of that used in [6] to determine $\sigma(T_\phi)$ for $\phi$ in $C$.

In §3 we indicate how to extend our results to determine the index of a certain class of vector-valued Toeplitz operators (or systems of Toeplitz operators). In this we generalize certain of the results of Gohberg and Krein in [7]. We conclude by describing how our results can be applied to the study of Wiener-Hopf operators both in the scalar case and the vector-valued case using the isomorphism exhibited in [5].

We only outline our proofs and complete details will appear elsewhere.

2. We begin by recalling some facts about Fredholm operators. Let $\mathcal{L}$ denote the algebra of bounded operators on $H^2$, $\mathcal{K}$ the uniformly closed two-sided ideal of compact operators in $\mathcal{L}$, and $\pi$ the homomorphism of $\mathcal{L}$ onto $\mathcal{L}/\mathcal{K}$. An operator $A$ in $\mathcal{L}$ is said to be a Fredholm operator if $A$ has a closed range and both a finite dimen-
sional kernel and cokernel. It is known [1] that this is equivalent to 
\( \pi(A) \) being an invertible element of \( \mathcal{E}/\mathcal{K} \). If \( A \) is a Fredholm operator, 
then the analytical index \( i_a(A) \) is defined \( i_a(A) = \dim [\ker A] - \dim [\ker A^*] \), where \( \ker(\ ) \) denotes the kernel.

One reason the notion of index is important for determining the 
invertibility of Toeplitz operators is the following fact proved by 
Coburn [2].

**Lemma 1.** For \( \phi \) in \( L^\infty \) either \( \ker T_\phi = (0) \) or \( \ker T_\phi^* = (0) \).

Thus, if \( T_\phi \) is known to be a Fredholm operator, then \( T_\phi \) is invertible if and only if \( i_a(T_\phi) = 0 \). We shall show for \( \phi \) in \( H^\infty + C \) that \( T_\phi \) is a Fredholm operator if and only if \( \phi \) is an invertible element of the algebra \( H^\infty + C \). That \( H^\infty + C \) is an algebra is a result due to Sarason [9] which we state as a lemma.

**Lemma 2.** The linear span of \( H^\infty \) and \( C \) is a closed subalgebra of \( L^\infty \). Moreover, the maximal ideal space of \( H^\infty + C \) is the maximal ideal space of \( H^\infty \) with the unit disk removed.

Sarason shows in [9] that the linear span of \( H^\infty \) and \( C \) is closed. The observation that the “closure” of \( H^\infty + C \) coincides with the closed subalgebra of \( L^\infty \) generated by \( H^\infty \) and \( \mathbb{Z} \) allows us to conclude that \( H^\infty + C \) is an algebra and to identify its maximal ideal space. The latter is a special case of the following proposition which is itself of interest. Let \( X \) be a compact Hausdorff space and \( \mathcal{A} \) be a uniformly closed subalgebra of the space \( C(X) \) of continuous complex functions on \( X \) which separates points and contains the constants. Let \( \phi \) be a function on \( X \) having modulus one in \( C(X) \) and let \( \mathcal{A}(\phi) \) denote the closed subalgebra of \( C(X) \) generated by \( \mathcal{A} \) and \( \phi \). Then the maximal ideal space for \( \mathcal{A}(\phi) \) is obtained from that of \( \mathcal{A} \) by deleting the open set on which the Gelfand transform of \( \phi \) has modulus less than one.

Now let \( \mathcal{G} \) denote the uniformly closed subalgebra of \( \mathcal{E} \) generated by the operators \( T_\phi \) with \( \phi \) in \( H^\infty + C \). Note that \( \mathcal{G} \) is not a \( C^* \)-algebra.

**Lemma 3.** The algebra \( \mathcal{G} \) contains \( \mathcal{K} \) as a two-sided ideal and \( \mathcal{G}/\mathcal{K} \) is isometrically isomorphic to \( H^\infty + C \).

**Proof.** Since \( \mathcal{G} \) contains the \( C^* \)-algebra generated by the unilateral shift of multiplicity one, it follows from [3] that \( \mathcal{G} \) contains \( \mathcal{K} \) and \( \mathcal{K} \) is an ideal in any algebra of \( \mathcal{E} \) containing it. If \( p \) and \( q \) are trigonometric polynomials and \( \psi \) and \( \zeta \) are functions in \( H^\infty \), then a straightforward computation shows that the commutator of \( T_{\psi + p} \) and \( T_{\zeta + q} \) is compact. Thus the linear span of the operators of the form \( T_\phi + K \), where \( \phi \) is in \( H^\infty + C \) and \( K \) is in \( \mathcal{K} \), is an algebra. That it is in fact a
closed algebra follows from the inequality \( \| T_\phi + K \| \geq \| T_\phi \| \) proved in [2]. Therefore, \( \mathcal{G}/\mathcal{K} \) is commutative and the mapping \( T_\phi + K \leftrightarrow \phi \) is an isometrical isomorphism of \( \mathcal{G}/\mathcal{K} \) onto \( H^\infty + C \).

**Corollary.** If \( \phi \) is in \( H^\infty + C \), then \( T_\phi - \lambda \) is a Fredholm operator if and only if \( \phi - \lambda \) is an invertible element of \( H^\infty + C \).

**Proof.** If \( \phi - \lambda \) is an invertible element of \( H^\infty + C \), then it follows from the preceding lemma that \( T_\phi - \lambda \) is a Fredholm operator. Conversely, if \( T_\phi - \lambda \) is a Fredholm operator, then \( \pi (T_\phi - \lambda) \) is an invertible element of \( \mathcal{E}/\mathcal{K} \) and we must show that its inverse is in \( \mathcal{G}/\mathcal{K} \). This can be shown for a \( \phi \) in \( H^\infty \) and the problem for an arbitrary \( \phi \) in \( H^\infty + C \) is solved by approximating \( \phi \) by a function of the form \( z^{-n} \psi \) where \( \psi \) is in \( H^\infty \).

The preceding result determines when \( T_\phi - \lambda \) is a Fredholm operator. This combined with Lemma 1 will enable us to determine \( \sigma (T_\phi) \) when we have some effective method of determining the index of \( T_\phi - \lambda \). If \( \phi \) is continuous, the index of \( T_\phi - \lambda \) is equal to the negative of the topological index \( i_t (\phi, \lambda) \) of the curve determined by \( \phi \) with respect to \( \lambda \) (cf. [6]). In the case at hand we use the index of the curves \( \phi (re^{i\theta}) \) where \( \phi \) is the harmonic extension of \( \phi \) to the interior of \( D \). To this end we need to relate the invertibility of \( \phi \) in \( H^\infty + C \) to the function \( \phi \) on \( D \).

**Lemma 4.** A necessary and sufficient condition that \( \phi \) in \( H^\infty + C \) be invertible is that \( \phi^{-1} \) be in \( L^\infty \) and for each \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( \| \phi (re^{i\theta}) \| \geq 1/\| \phi^{-1} \|_\infty - \epsilon \) for \( 1 > r \geq 1 - \delta \).

**Proof.** We again approximate \( \phi \) by a function of the form \( z^{-n} \psi \) with \( \psi \) in \( H^\infty \) and analyze the inner and outer factors of \( \psi \).

**Lemma 5.** If \( T_\phi - \lambda \) is a Fredholm operator, then \( i_t (T_\phi - \lambda) = -\lim_{R \to 1^-} i_t (\phi (Re^{i\theta}), \lambda) \).

**Proof.** From the Corollary and Lemma 4 it follows that for \( T_\phi - \lambda \) a Fredholm operator there exists \( 0 < R < 1 \) so that \( \phi (re^{i\theta}) \neq \lambda \) for \( 1 > r \geq R \). Since \( \phi \) is continuous on \( D \) we then have that \( i_t (\phi (re^{i\theta}), \lambda) \) is constant for \( r \geq R \) so that the limit exists. The proof is now accomplished with the same technique used in the preceding proof.

We can now determine the spectrum of \( T_\phi \) for \( \phi \) in \( H^\infty + C \).

**Theorem.** For \( \phi \) in \( H^\infty + C \) we have \( T_\phi \) is invertible if and only if

\[
\lim_{R \to 1^-} \inf_{0 \leq \theta < 2\pi; R \leq r < 1} | \phi (re^{i\theta}) | = \eta > 0
\]

and
COROLLARY. For $\phi$ in $H^\omega + C$ we have

$$\sigma(T_\phi) = \left\{ \lambda \left| \lim_{R \to 1^-} \inf_{0 < \theta < 2\pi, R \leq r < 1} |\hat{\phi}(re^{i\theta}) - \lambda| = 0 \right\} \cup \left\{ \lambda \left| \lim_{R \to 1^-} i_t(\hat{\phi}(re^{i\theta}), \lambda) \neq 0 \right\} \right.$$.

We make several comments before continuing to the vector valued case. Firstly, although the statement of the Theorem makes sense for an arbitrary $\phi$ in $L^\omega$ the Theorem is not valid in this generality. Secondly, $\lim_{r \to 0^+} i_t(\hat{\phi}(re^{i\theta}), \lambda) = 0$ does not imply that $T_\phi$ is invertible even for $\phi$ in $H^\omega$. Thirdly, Widom has shown that $\sigma(T_\phi)$ is connected for $\phi$ in $L^\omega$ (cf. [11]). We remark that it follows from the Corollary to Lemma 3, Lemma 1 and the fact that the maximal ideal space of $H^\omega + C$ is connected [8] that $\sigma(T_\phi)$ is connected for $\phi$ in $H^\omega + C$. Lastly, using the identification of Wiener-Hopf operators with Toeplitz operators (cf. [5]), our Theorem can be used to determine the invertibility of a certain class of Wiener-Hopf operators.

We now describe the extension of our results to the vector valued case. Let $\mathcal{E}$ be a finite dimensional Hilbert space and $\mathcal{L}(\mathcal{E})$ the ring of bounded operators on $\mathcal{E}$. Let $L^{2}_{\mathcal{E}}$ denote the Hilbert space of measurable $\mathcal{E}$-valued functions on $T$ having square integrable norm, $H^{2}_{\mathcal{E}}$ the corresponding Hardy space of functions in $L^{2}_{\mathcal{E}}$ which have zero negative Fourier coefficients, and $P$ the projection of $L^{2}_{\mathcal{E}}$ onto $H^{2}_{\mathcal{E}}$. Further, let $L^{2}_{\mathcal{E}(\mathcal{E})}$ denote the ring of bounded measurable $\mathcal{L}(\mathcal{E})$-valued functions on $T$ and $H^{2}_{\mathcal{E}(\mathcal{E})}$ the Hardy space of functions in $L^{2}_{\mathcal{E}(\mathcal{E})}$ with zero negative Fourier coefficients. For $\Phi$ in $L^{2}_{\mathcal{E}(\mathcal{E})}$ we define the Toeplitz operator $T_{\Phi}$ on $H^{2}_{\mathcal{E}}$ by $T_{\Phi}f = P(\Phi f)$, where $\Phi f$ denotes the pointwise product. Finally, let $C_{\mathcal{L}(\mathcal{E})}$ denote the space of continuous $\mathcal{L}(\mathcal{E})$-valued functions on $T$.

THEOREM. If $\Phi$ is in $H^{2}_{\mathcal{E}(\mathcal{E})} + C_{\mathcal{E}}$ then $T_{\Phi}$ is a Fredholm operator if and only if

$$\lim_{R \to 1^-} \inf_{0 < \theta < 2\pi, R \leq r < 1} |(\det \Phi)^{\wedge}(Re^{i\theta})| = \eta > 0$$

and this case

$$i_{\eta}(T_{\phi}) = - \lim_{R \to 1^-} i_t((\det \Phi)^{\wedge}(Re^{i\theta})).$$

PROOF. We briefly describe the changes necessary in the proof given for the scalar case. We define $\mathcal{F}$ as the closed subalgebra of $\mathcal{L}(H^{2}_{\mathcal{E}})$.
generated by the $T_\Phi$ for $\Phi$ in $H^*_\mathcal{E}(\theta) + C_{\mathcal{L}(\theta)}$ and show $\mathcal{G}$ contains as an ideal, the ring $\mathcal{K}$ of compact operators on $H^*_{\mathcal{G}}$. Further, we show that $\mathcal{G}/\mathcal{K}$ is isometrically isomorphic to the closed subalgebra $H^*_\mathcal{E}(\theta) + C_{\mathcal{L}(\theta)}$ of $L^*_\mathcal{E}(\theta)$. Again we show that $\pi(T_\Phi)$ is invertible in $\mathcal{G}/\mathcal{K}$ if and only if it is invertible in $\mathcal{L}/\mathcal{K}$. Thus we find that $T_\Phi$ is a Fredholm operator if and only if $\det(\Phi)$ is an invertible element of $H^* + C$. Lastly, the index of $T_\Phi$ is computed using the fact that the determinant defines an isomorphism from the first homotopy group of the general linear group for $\mathcal{C}$ onto the first homotopy group of the space of nonzero complex numbers.

Again using the isomorphism exhibited in [5] we can identify the operator $T_\Phi$ with a matrix valued Wiener-Hopf operator. In this context Gohberg and Kreín proved the preceding theorem for $\Phi$ in a certain subset of $C_{\mathcal{L}(\theta)}$.

Complete details will appear elsewhere along with extensions of the preceding results to the case of an infinite dimensional space $\mathcal{C}$ as well as to Toeplitz operators defined on certain other Banach spaces.

BIBLIOGRAPHY


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