TWO TYPES OF LOCALLY COMPACT RINGS

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Here we shall present structure theorems for two types of commutative locally compact rings with identity. The first is for rings satisfying a rather stringent topological condition, namely, that there exist an invertible, topologically nilpotent element. An analysis of such rings requires basic theorems of commutative algebra and, in particular, a decomposition theorem for total quotient rings of one-dimensional Macaulay rings. One consequence of the structure theorem is the determination of necessary and sufficient conditions for a locally compact ring with identity to be the topological direct product of topological algebras over indiscrete locally compact fields.

The second is a theorem classifying all compatible metrizable locally compact topologies on a ring satisfying very stringent algebraic conditions, namely, that the ring be a special principal ideal ring in the sense of Zariski and Samuel [6, p. 245] of either zero or prime characteristic. For this investigation we require a theorem concerning finite-dimensional, locally compact, metrizable vector spaces over discrete fields, which shows that, in a certain sense, such spaces are not too remote from finite-dimensional vector spaces over indiscrete locally compact fields.

1. Commutative locally compact rings having an invertible, topologically nilpotent element. We recall that a local ring is a commutative ring with identity that has only one maximal ideal, and that the natural topology of a local noetherian ring is obtained by declaring the powers of its maximal ideal a fundamental system of neighborhoods of zero. Moreover, a local noetherian ring is compact for its natural topology if and only if it is complete and its residue field is finite. If \( A \) is a compact ring that algebraically is a local noetherian ring, then the topology of \( A \) is its natural topology [4, Theorem 4]. We recall also that a one-dimensional local noetherian ring is a Macaulay ring if and only if its maximal ideal is not an associated prime ideal of the zero ideal [7, p. 397].

Let \( B \) be a one-dimensional Macaulay ring topologized with its natural topology, and let \( m \) be its maximal ideal, \( p_1, \ldots, p_n \) the (isolated) prime ideals of the zero ideal. The complement of \( p_1 \cup \ldots \cup p_n \) is the set of cancellable elements of \( B \). Let \( A \) be the total quotient

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ring of \( B \), topologized by declaring the neighborhoods of zero in \( B \) to be a fundamental system of neighborhoods of zero in \( A \); this topology we call the \( B \)-topology on \( A \). To show that \( A \) is a topological ring, it suffices to show that \( x \mapsto b^{-1}x \) is continuous at zero for any cancellable \( b \in B \); this is accomplished by observing that \( m \) is the only prime ideal of \( Bb \). Any element of \( m \) not belonging to \( p_1 \cup \ldots \cup p_n \) is an invertible, topologically nilpotent element of \( A \).

We shall say that a local noetherian ring is aligned if the prime ideals, ordered by inclusion, form a chain. Thus a one-dimensional aligned local noetherian ring has precisely two proper prime ideals, one contained in the other. The decomposition theorem needed is the following:

**Theorem 1.** If \( A \) is the total quotient ring of a one-dimensional Macaulay ring \( B \) and if \( A \) is topologized by the \( B \)-topology, then \( A \) is the topological direct product of ideals \( A_1, \ldots, A_n \), where each \( A_k \) is the total quotient ring of a one-dimensional aligned Macaulay ring \( B_k \) and is topologized by the \( B_k \)-topology.

A semilocal ring is a commutative ring with identity that has only finitely many maximal ideals. A Cohen algebra is a local algebra over a field whose maximal ideal has codimension one. Our structure theorem for commutative locally compact rings having an invertible, topologically nilpotent element is the following:

**Theorem 2.** Let \( A \) be a commutative locally compact ring with identity. The following statements are equivalent:

1°. \( A \) contains an invertible element that is topologically nilpotent.

2°. \( A \) is semilocal, and none of its maximal ideals is open.

3°. \( A \) is the topological direct product of a sequence \( (A_k)_{1 \leq k \leq n} \) of ideals where each \( A_k \) is either a locally compact finite-dimensional Cohen algebra over the topological field of real or complex numbers or the topological quotient ring of a compact one-dimensional aligned Macaulay ring.

**Outline of Proof.** We first assume that \( A \) is totally disconnected and satisfies 1°. By a lemma of Kaplansky [3, Lemma 5], \( A \) contains a compact open subring \( B \) that contains the identity element of \( A \). By Kaplansky's characterization of compact semisimple rings [2, Theorem 16], the existence of an invertible, topologically nilpotent element implies that the radical \( R \) of \( B \) is open, so \( B/R \) is the cartesian product of finitely many finite fields. Raising idempotents from \( B/R \) to \( B \) and then using them to decompose \( A \), we conclude that \( A \) is the topological direct product of ideals \( A_1, \ldots, A_m \) where
each $A_i$ contains a compact, open, local subring $B_i$. Once again the hypothesis that $A_i$ has an invertible topologically nilpotent element implies that all the powers of the radical of $B_i$ are open, so by a theorem of Kaplansky [2, Theorem 20], $B_i$ is a local noetherian ring. It is easy to see, in fact, that $B_i$ is a one-dimensional Macaulay ring and that $A_i$ is the total quotient ring of $B_i$ equipped with the $B_i$-topology. An application of Theorem 1 to each $A_i$ shows that 3° holds and, in particular, that the radical of $A$ is nilpotent. The general case is now established by use of the Pontryagin-van Kampen theorem on commutative, locally compact, connected groups, the nilpotence of the radical in the totally disconnected case, and the fact, proved by using standard theorems concerning finite-dimensional topological vector spaces over locally compact fields, that a locally compact local ring whose maximal ideal is nilpotent but not open is either connected or totally disconnected [5, Lemma 7].

**Theorem 3.** Let $A$ be a commutative locally compact ring with identity. The following statements are equivalent:

1°. $A$ contains an invertible element that is topologically nilpotent, and the additive order of each element of $A$ is either infinite or a square-free integer.

2°. $A$ is semilocal, none of its maximal ideals is open, and the additive order of each element of $A$ is either infinite or a square-free integer.

3°. $A$ is the topological direct product of topological algebras over indiscrete locally compact fields.

4°. $A$ is the topological direct product of finitely many finite-dimensional Cohen algebras over indiscrete locally compact fields.

**Outline of Proof.** To show that 1° implies 4°, it suffices by Theorem 2 to consider the case where $A$ is the total quotient ring of a one-dimensional aligned compact Macaulay ring $B$, equipped with the $B$-topology. Then $A$ is local, and its maximal ideal $m$ is closed, not open, and nilpotent. Consequently, $A$ and $A/m$ have the same characteristic. It follows easily that $A$ contains an indiscrete topological subfield $K$ that is the quotient field of a principal ideal domain $D$ and that the open $D$-submodules of $K$ form a fundamental system of neighborhoods of zero in $K$. We may therefore apply Corrol's theorem [1, Theorem 3] to conclude that the completion of $K$ is a locally compact field. A modification of a proof of I. S. Cohen's theorem on complete equicharacteristic local rings [7, pp. 304–306] enables us to replace this field by a locally compact subfield that is canonically mapped onto $A/m$ [5, Lemma 5].
COROLLARY. Let $A$ be a locally compact ring with identity. The following statements are equivalent:

1°. The center of $A$ contains an invertible element that is topologically nilpotent, and the additive order of each element of $A$ is either infinite or a square-free integer.

2°. $A$ is the topological direct product of finitely many topological algebras over indiscrete locally compact fields.

2. Locally compact metrizable special principal ideal rings. A special principal ideal ring [6, p. 245] is a principal ideal ring that has only one proper prime ideal, and that ideal is nilpotent. If $A$ is a special principal ideal ring whose characteristic is either zero or a prime, then by I. S. Cohen’s theorem $A$ is an algebra over a field $K$ that has a basis $1, c, c^2, \ldots, c^s$, where $c^s = 0$. Suppose that $K$ admits an indiscrete locally compact topology compatible with its field structure, and let $r \in [0, s - 1]$. Then $Ac^r$ is the finite-dimensional subspace generated by $c^r, \ldots, c^{s-1}$ and hence admits a unique topology making it a Hausdorff vector space over $K$. We topologize $A$ by declaring the neighborhoods of zero in $Ac^r$ to be neighborhoods of zero in $A$.

It is not difficult to verify that $A$, so topologized, is a topological ring (though if $r > 0$, $A$ is a topological algebra over $K$ only if $K$ is given the discrete topology). This topology depends only on the topological field $K$ and the numbers $r$ and $s$, so we shall call it the $(K, r, s)$-topology. To show that every compatible locally compact metrizable topology on $A$ is a $(K, r, s)$-topology, we require the following theorem:

**Theorem 4.** Let $E$ be a totally disconnected, finite-dimensional, locally compact, metrizable vector space [algebra] over a discrete field $K$, and let $L = \{x \in E: \text{either } x = 0 \text{ or } Kx \text{ is indiscrete} \}$. Then $L$ is an open subspace [open ideal] of $E$, and $L$ is the topological direct sum of subspaces [ideals of $E$] $E_1, \ldots, E_n$, where for each $i \in [1, n]$, the locally compact group [ring] $E_i$ admits the structure of finite-dimensional topological vector space [algebra] over an indiscrete locally compact field $F_i$ under a scalar multiplication satisfying $\alpha.(\mu x) = \mu(\alpha.x)$ [and also $\alpha.(xy) = (\alpha.x)y$, $\alpha.(yx) = y(\alpha.x)$] for all $\alpha \in F_i$, $\mu \in K$, $x \in E_i$ [and $y \in E$]. If $N$ is any algebraic supplement of $L$, then $N$ is discrete, and $E$ is the topological direct sum of $E_1, \ldots, E_n, N$.

Other than elementary facts, the proof depends only on the following three theorems: (1) The Baire Category Theorem (to show that if $E$ is indiscrete, then $K$ is uncountable); (2) There exist no
nonzero compact metrizable vector spaces over an uncountable dis­
tcrete field (a consequence of the theory of characters); (3) The Open
Mapping Theorem for separable, metrizable, locally compact groups.
An analogue of Theorem 4 also holds for connected, locally compact,
metrizable vector spaces and algebras over discrete fields.

**Theorem 5.** Let $A$ be an indiscrete, locally compact, metrizable,
special principal ideal ring whose characteristic is either zero or a prime,
and let $m$ be the maximal ideal of $A$. The topology of $A$ is then the
$(K, r, s)$-topology, where $K$ is an indiscrete locally compact field that
algebraically is a subfield of $A$ and is mapped canonically onto $A/m$,
where $r$ is the largest integer such that $m^r$ is open, and where $s$ is the index
of nilpotence of $m$.

**Outline of Proof.** From the existence of a coefficient field, The­
orem 4, and the fact that a field cannot admit both a connected and a
totally disconnected locally compact topology compatible with its
field structure, it follows that $m^r$ with its induced topology is a con­
nected or totally disconnected, metrizable, locally compact finite-
dimensional algebra over an indiscrete locally compact field $F$. Using
again the special case of I. S. Cohen’s theorem for equicharacteristic
local rings having a nilpotent maximal ideal, we find a subfield $K_0$
of $A$ that is algebraically isomorphic to $F$ and acts on $m^r$ as $F$ does.
Transferring the topology of $F$ to $K_0$ and applying [5, Lemma 5],
we obtained the desired conclusion.

**References**

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