A NOTE ON THE STRONG LAW OF LARGE NUMBERS

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1. Introduction. Let \( \{X_k\} \) denote a sequence of independent, identically distributed (i.i.d.) random variables. Let

\[
S_n = \sum_{k=1}^{n} X_k \quad (n = 1, 2, \ldots).
\]

A long standing problem in probability theory has been to find necessary and sufficient conditions on the distribution function of \( X_k \) in order that \( n^{-1}S_n \) converge almost surely to plus infinity. The purpose of this paper is to exhibit such conditions.

2. Theorem. Let \( \{X_k\} \) denote a sequence of i.i.d. random variables with common characteristic function \( \phi \). Then \( n^{-1}S_n \to +\infty \) a.s. if and only if

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{e^{iu\theta} - 1}{iu} \log \left\{ 1 - \frac{e^{-iu\phi(u)}}{1 + u^2} \right\}^{-1} du
\]

is finite for each \( a > 0 \).

The proof of the theorem is based on the following lemma.

Lemma. Let \( \{X_k\} \) denote a sequence of i.i.d. random variables. Then

(a) \( n^{-1}S_n \to +\infty \) a.s. if and only if, for each \( a > 0 \),

\[
\sum_{n=1}^{\infty} n^{-1}P(S_n < an) < \infty.
\]

Proof of the Lemma. We first show that (a) implies (b). Suppose there exists an \( a > 0 \) such that \( \sum n^{-1}P(S_n < an) = \infty \). Let

\[
T_n = \sum_{k=1}^{n} (a - X_k).
\]

Then \( \sum n^{-1}P(T_n > 0) = \infty \), and, by a theorem of Spitzer [2] it follows that \( \lim \sup T_n = \infty \) a.s. However, \( \lim n^{-1}S_n = \infty \) a.s. certainly implies that \( \lim \sup T_n = -\infty \) a.s. Thus (b) holds.

Conversely, suppose that (b) holds. Then, for each \( a > 0 \),

\[
\sum n^{-1}P(S_n - na < 0) < \infty
\]

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and consequently, from the same work of Spitzer, we have that
\[
\max_{k=1} \max (ka - S_k)^+ < \infty \text{ a.s.}
\]
for all \(a > 0\). This clearly implies that \(\lim n^{-1} S_n = \infty \) a.s., and therefore (a) holds.

Before proceeding to the proof of the theorem, we note that previous work (e.g. Derman and Robbins [1]) giving sufficient conditions that \(\lim n^{-1} S_n = \infty \) a.s. follows quickly from the above lemma.

**Proof of the Theorem.** Let \(\{Y_k\}\) denote a sequence of i.i.d. random variables, each with characteristic function \((1+u^2)^{-1}\), and, further, let \(\{Y_k\}\) be independent of \(\{X_k\}\). Write \(Z_k = X_k + Y_k\) and
\[
W_n = \sum_{k=1}^{n} Z_k \quad (n = 1, 2, \ldots).
\]
Since \(Y_k\) has expectation zero, \(n^{-1} S_n \to \infty \) a.s. if and only if \(n^{-1} W_n \to \infty \) a.s.

By means of a well-known inversion formula, we have that
\[
P(an - b \leq W_n < an) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iub} \left( \frac{e^{iub} - 1}{iu} \right) \left( \frac{\phi(u)}{1 + u^2} \right)^n du.
\]
(Note that \(\{\phi(u)/(1+u^2)\}^n\) is integrable and that \(W_n\) has an absolutely continuous distribution function.)

Therefore,
\[
\sum_{n=1}^{\infty} n^{-1} P(an - b \leq W_n < an) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iub} \left( \frac{e^{iub} - 1}{iu} \right) \log \left( 1 - \frac{e^{-iua}\phi(u)}{1 + u^2} \right)^{-1} du,
\]
the interchange of sum and integral being justified, since
\[
\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{-iun} \left( \frac{e^{iub} - 1}{iu} \right) \left( \frac{\phi(u)}{1 + u^2} \right)^n du
\leq b \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\phi(u)}{1 + u^2} \right)^n \right\} du
\leq b \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{1 + u^2} \right)^n \right\} du
= b \int_{-\infty}^{\infty} \log \left( 1 + \frac{1}{u^2} \right) du
< \infty.
\]
From the Monotone Convergence Theorem, it follows that

$$\sum_{n=1}^{\infty} n^{-1}P(W_n < an) = \lim_{b \to \infty} \sum_{n=1}^{\infty} n^{-1}P(an - b \leq W_n < an)$$

$$= \lim_{b \to \infty} \int_{-\infty}^{\infty} \frac{e^{ibu} - 1}{iu} \log \left( 1 - \frac{e^{-iua}\phi(u)}{1 + u^2} \right)^{-1} du.$$

By the Riemann-Lebesgue lemma,

$$\lim_{b \to \infty} \int_{|u| > 1} \frac{e^{ibu} - 1}{iu} \log \left( 1 - \frac{e^{-iua}\phi(u)}{1 + u^2} \right)^{-1} du$$

exists and is finite. It follows that (1) is finite for each $a > 0$ if and only if $\sum n^{-1}P(W_n < an)$ is finite for each $a > 0$. The latter condition is equivalent to $n^{-1}W_n \to \infty$ a.s. which is, in turn, equivalent to $n^{-1}S_n \to \infty$ a.s. This completes the proof.

REFERENCES


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