THE STRUCTURE OF TORSION ABELIAN GROUPS
GIVEN BY PRESENTATIONS

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Communicated by R. S. Pierce, March 1, 1968

Let $F_X$ denote the free abelian group freely generated by the set $X$, and let $R$ be a subset of $F_X$. With $[R]$ denoting the subgroup of $F_X$ generated by $R$, set

$$G(X, R) = F_X/[R],$$

i.e., $G(X, R)$ is that abelian group generated by $X$ and subject only to the relations $r = 0$ all $r \in R$.

If each of the elements in $R$ involves only one generator in $X$, then $G(X, R)$ is a direct sum of cyclic groups. On the other hand, if $G$ is any abelian group, then $G \cong G(X, R)$, where each element in $R$ involves at most three generators in $X$; indeed this isomorphism results if we take $X = G$ and $R$ equal to the set of all elements in $F_G$ of the form $x+y-z$, where $z = x+y$ in $G$.

Our purpose here is to investigate the structure of the group $G(X, R)$ in the intermediate case when each of the elements of $R$ involves at most two generators, and $G(X, R)$ is a torsion group. We can evidently restrict our attention to $p$-groups, and in this case it is easily seen that $G(X, R) \cong G(X', R')$, where each element in $R'$ is of one of the forms

$p^nx$ or $p^nx - y$.

This leads us to the following definition. Let $X$ be a set, $V$ be a subset of the set of ordered pairs $(x, y)$ with $x, y \in X$, $u$ be a map of $X$ to the nonnegative integers, and $v$ be a map of $V$ to the nonnegative integers. By $G(X, V, u, v)$ we mean that abelian group generated by $X$ and subject only to the relations

$p^{u(x)}x = 0$ all $x \in X$,  
$p^{v(x,y)}x = y$ all $(x, y) \in V$.

We say that an abelian $p$-group $G$ is a $T$-group if $G \cong G(X, V, u, v)$ for some $(X, V, u, v)$.

1 This work was supported in part by NSF Grants GP 7252 and GP 5497.
One property of $T$-groups is clear: the direct sum of a family of $T$-groups is again a $T$-group. Every divisible $p$-group is certainly a $T$-group, and the reduced part of a $T$-group is again a $T$-group.

Before stating our main results concerning these groups, let us recall a few basic definitions. Let $G$ be any reduced abelian $p$-group. Define the subgroups $p^\alpha G$ for each ordinal $\alpha$ as usual by the rules:

$$p^0 G = G; \quad p^\alpha G = \{px | x \in p^{\alpha-1} G\} \text{ if } \alpha - 1 \text{ exists}; \quad p^\alpha G = \cap_{\beta < \alpha} p^\beta G \text{ if } \alpha \text{ is a limit ordinal.}$$

Since $G$ is reduced, there is a first ordinal $\lambda$, called the length of $G$, such that $p^\lambda G = 0$. For each ordinal $\alpha$ we set

$$f_\alpha(\alpha) = \text{rank } p^\alpha G \cap G[p]/p^{\alpha+1} G \cap G[p],$$

where $G[p] = \{x \in G | px = 0\}$, and we call the cardinal number $f_\alpha(\alpha)$ the $\alpha$th Ulm invariant of $G$. Finally we let $\omega$ denote the first infinite ordinal and $\Omega$ denote the first uncountable ordinal.

The description of $T$-groups is now accomplished by the following theorems.

(A) If $G$ and $H$ are reduced $T$-groups, then $G$ and $H$ are isomorphic if and only if $f_\alpha(\alpha) = f_\beta(\beta)$ for each ordinal $\alpha$.

(B) Let $f$ be a map of an ordinal $\lambda$ to a set of cardinal numbers. Then there exists a reduced $T$-group $G$ of length $\lambda$ such that $f_\alpha(\alpha) = f(\alpha)$ for each $\alpha < \lambda$, if and only if $f$ satisfies the following conditions:

(i) $\lambda = \sup \{\alpha + 1 | f(\alpha) \neq 0\}$;

(ii) if $\alpha$ is a limit ordinal such that $\alpha + \omega < \lambda$, and $0 \leq \eta < \omega$, then

$$\sum_{\alpha + \eta + \beta < \alpha + \omega} f(\beta) \geq \sum_{\alpha + \eta + \beta < \lambda} f(\beta).$$

(C) A reduced $p$-group $G$ is a direct sum of countable groups if and only if $G$ is a $T$-group of length at most $\Omega$.

When specialized to countable $p$-groups, (A) and (C), of course, reduce to Ulm's Theorem, and in the case of direct sums of countable groups they reduce to the theorem of Kolettis [2]. Our results are not independent of Ulm's Theorem, however, since it is used to establish (C). The proofs of (A), (B) and (C) will appear elsewhere.

Actually $T$-groups have been studied before in a different guise. In [3], Nunke defines a reduced $p$-group $G$ to be totally projective if

$$p^\alpha \text{Ext}(G/p^\alpha G, A) = 0$$

for all ordinals $\alpha$ and every group $A$, and he obtains a number of properties of these groups. Quite recently Hill [1] has announced that two totally projective groups with the same Ulm invariants are isomorphic. Now it is easily verified that if $G$ is a reduced $T$-group and $\alpha$ is
an ordinal, then both \( p^\alpha G \) and \( G/p^\alpha G \) are T-groups. Moreover, (A) and (B) yield that a T-group whose length is a limit ordinal is a direct sum of T-groups of smaller length. These last two facts, in conjunction with [3, 2.6], imply that every reduced T-group is totally projective. On the other hand, if \( H \) is a totally projective group, then the function \( f_H \) necessarily satisfies condition (ii) of (B). Consequently there is a reduced T-group \( G \) having the same Ulm invariants as \( H \), and Hill's theorem guarantees that \( G \) and \( H \) are isomorphic. Thus a reduced abelian \( p \)-group is totally projective if and only if it is a T-group.

The foregoing results further provide a characterization of the class of all reduced T-groups in terms of certain natural group-theoretic properties. Let \( \mathcal{K} \) be a class of reduced abelian \( p \)-groups. Then \( \mathcal{K} \) coincides with the class of all reduced T-groups if and only if \( \mathcal{K} \) has the following properties: (1) \( \mathcal{K} \) is closed under isomorphism; (2) \( \mathcal{K} \) is closed under direct sums; (3) if \( G \in \mathcal{K} \) and the length of \( G \) is a limit ordinal, then \( G \) is a direct sum of groups in \( \mathcal{K} \) of smaller length; (4) for each \( p \)-group \( G \) and each ordinal \( \alpha, G \in \mathcal{K} \) if and only if both \( G/p^\alpha G, p^\alpha G \in \mathcal{K} \); (5) if an abelian \( p \)-group \( G \) has no elements of infinite height, i.e., \( p^\infty G = 0 \), then \( G \in \mathcal{K} \) if and only if \( G \) is a direct sum of cyclic groups. Thus the class of all reduced T-groups is the smallest class of reduced \( p \)-groups that has properties (1)–(4) and contains the finite groups.

References


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