NOTE ON PRINCIPAL $S^n$-BUNDLES

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In this note we construct two principal $S^n$-bundles whose total spaces $E_\alpha$, $E_\beta$ are closed smooth manifolds having the properties

(i) $E_\alpha$, $E_\beta$ are of different homotopy types, $E_\alpha \not\approx E_\beta$;
(ii) $E_\alpha \times S^3$, $E_\beta \times S^3$ are diffeomorphic.

The method of construction is a modified dual of that employed in [1] to demonstrate the failure of wedge-cancellation.

Let $a, b \in \pi_n(S^3)$, let $B$ be the classifying space for $S^3$, and let $\alpha, \beta \in \pi_{n+1}(B)$ be the elements corresponding to $a$, $b$ respectively. Let $\pi_\alpha: E_\alpha \to S^{n+1}$, $\pi_\beta: E_\beta \to S^{n+1}$ be the bundle projections induced by $\alpha, \beta$.

**Theorem 1.** $E_\alpha \simeq E_\beta$ if and only if $\beta = \pm \alpha$ (equivalently, $b = \pm a$).

**Proof.** Sufficiency is obvious, so we suppose $E_\alpha \simeq E_\beta$ and seek to prove $\beta = \pm \alpha$. If $n \leq 2$, the assertion is trivial. Now there are cell-decompositions

$$E_\alpha = S^8 \cup_a e^{n+1} \cup e^{n+4}, \quad E_\beta = S^8 \cup_b e^{n+1} \cup e^{n+4}.$$  

Thus if $n = 3$, $a$ and $b$ are integers and $H_3(E_\alpha) = \mathbb{Z}_{|a|}$, $H_3(E_\beta) = \mathbb{Z}_{|b|}$, whence $|a| = |b|$. We assume now that $n \geq 4$ and let $h: E_\alpha \simeq E_\beta$. We may suppose $h(S^3) \subseteq S^3$ and then $h|S^3$ is of degree $\pm 1$. From the exact homotopy sequence we infer that $h$ induces an isomorphism $\pi_{n+1}(E_\alpha, S^3) \cong \pi_{n+1}(E_\beta, S^3)$; these groups are cyclic infinite, generated by $i_\alpha, i_\beta$ say, so that $h(i_\alpha) = \pm i_\beta$. We have a commutative square

$$\pi_{n+1}(E_\alpha, S^3) \xrightarrow{h_*} \pi_{n+1}(E_\beta, S^3) \quad \downarrow \partial \quad \downarrow \partial
$$

$$\pi_n(S^3) \cong \pi_n(S^3)$$

where the bottom isomorphism is multiplication by $\pm 1$, $\partial(i_\alpha) = a$, $\partial(i_\beta) = b$. Thus $\pm b = \pm a$ or $\beta = \pm \alpha$.

Let $E_\alpha \beta \to E_\alpha$ be induced from $\pi_\alpha$ by $\pi_\alpha: E_\alpha \to S^{n+1}$, and let $E_\beta \alpha \to E_\beta$ be defined similarly.

**Theorem 2.** $E_\alpha \beta = E_\beta \alpha$. Moreover, $E_\alpha \beta$ is equivalent to $E_\alpha \times S^3$ if $\beta \circ \pi_\alpha = 0$ and $E_\beta \alpha$ is equivalent to $E_\beta \times S^3$ if $\alpha \circ \pi_\beta = 0$.

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1 Here and later we deliberately confuse maps and homotopy classes.
Thus it remains to choose $\alpha, \beta$ so that $\beta \circ \pi_a = 0$, $\alpha \circ \pi_\beta = 0$, and $\beta \neq \pm \alpha$. We now assume $n \geq 4$.

We construct the Puppe sequence for the inclusion $S^n \xrightarrow{i} E_a$, namely,

$$S^3 \xrightarrow{i} E_a \xrightarrow{p} Y \xrightarrow{u} S^4 \rightarrow \cdots,$$

where $Y = S^{n+1} \cup e^{n+4}$. Plainly $\pi_a = q \circ p$ for $q: Y \rightarrow S^{n+1}$, so that

$$0 = \alpha \circ \pi_a = \alpha \circ q \circ p,$$

whence

$$\alpha \circ q = \delta \circ u,$$

for some $\delta: S^4 \rightarrow B$.

We have the 'fibration'

$$S^7 \xrightarrow{h} S^4 \xrightarrow{e} B,$$

where $h$ is the Hopf map and $e$ generates $\pi_4(B) \cong \mathbb{Z}$. Then, since $Y$ is a double suspension,

$$u = h \circ v + u',$$

for some $v: Y \rightarrow S^7$,

where $u'$ is a suspension. Moreover, $\delta = me$ for some integer $m$ and, for any integer $s$,

$$se \circ h = \frac{s(s-1)}{2} [e, e],$$

where $[\ ,\ ]$ denotes the Whitehead product. Thus, for any integer $l$,

$$l \alpha \circ q = l(\alpha \circ q)$$

$$= l(d \circ u)$$

$$= l(d \circ h \circ v + d \circ u')$$

$$= d \circ h \circ lv + ld \circ u',$$

since $u'$ is a suspension.

On the other hand

$$ld \circ u = lme \circ h \circ v + ld \circ u'$$

$$= \frac{lm(lm - 1)}{2} [e, e] \circ v + ld \circ u'.$$

Now $12 [e, e] = 0$. Thus if we choose $l$ so that $lv = 0$ and $l \equiv 0 \pmod{24}$, then

$$l \alpha \circ q = ld \circ u' = ld \circ u.$$
But then $l \alpha \circ \pi_a = l \alpha \circ \pi \circ \rho = 0$. Now we have an exact sequence

$$\pi_{n+4}(S^7) \to \pi(Y, S^7) \to \pi_{n+1}(S^7);$$

thus if $r_1$ is the exponent of $\pi_{n+4}(S^7)$ and $r_2$ is the exponent of $\pi_{n+1}(S^7)$ we may take

$$l_0 = \text{l.c.m.}(r_1 r_2, 24)$$

and we have

**Theorem 3.** If $l_0 \mid l$ and $\beta = l \alpha$, then $\beta \circ \pi_a = 0$.

Naturally we may interchange the roles of $\alpha, \beta$ here; $l_0$ remains unchanged. We take $n = 17$; then $\pi_{17}(S^9) = \mathbb{Z}_{24}$ (see [2]) and we choose $a \in \pi_{17}(S^9)$ of order 5. From [2] we see that $r_1 = 24$, $r_2 = 504$, so we may certainly choose $l$ so that $l_0 \mid l$ and $l \equiv 2 \mod 5$. Thus if $\beta = 2\alpha$, $\beta \circ \pi_a = 0$. But then $b = 2a$ is of order 5 and $a = 3b$, and we may choose $l$ so that $l_0 \mid l$ and $l \equiv 3 \mod 5$. Thus we also have $\alpha \circ \pi_{17} = 0$. On the other hand $\beta \neq -a$, so that we have constructed the promised example, in which $E_a, E_\beta$ are principal $S^3$-bundles over $S^{18}$.

**References**