Problem 14 of Hanna Neumann's book [3] asks for a proof of a conjecture which is contradicted by the following

Theorem. If \( c \) is an integer greater than 2, then the variety \( \mathcal{N}_c \) of all nilpotent groups of class at most \( c \) is generated by its free group \( F_{c-1}(\mathcal{N}_c) \) of rank \( c-1 \) but not by its free group \( F_{c-2}(\mathcal{N}_c) \) of rank \( c-2 \).

In the terms of [3], this means that for \( c > 2 \) one has \( d(c) = c - 1 \) rather than \( d(c) = \lceil c/2 \rceil + 1 \) as suggested in Problem 14; correspondingly, [3, 35.35] is false for \( c = 5 \) and 6. (Professor Neumann has confirmed that her proofs were faulty.)

The theorem was suggested by Graham Higman's approach to nilpotent varieties of class \( c \) and prime exponent greater than \( c \), via the representation theory of the general linear groups [1]. In particular, he remarked that each critical group in such a variety can be generated by \( c - 1 \) elements (if \( c > 2 \)). Since \( F_c(\mathcal{N}_c) \) generates \( \mathcal{N}_c \) (cf. [3, 35.12]) and is residually of prime exponent (cf. Higman [2]), it follows easily that \( F_{c-1}(\mathcal{N}_c) \) generates \( \mathcal{N}_c \). It is not difficult to use Higman's method for confirming the second half of the theorem as well.

In this note we outline a proof which avoids the conceptual complexity of Higman's approach; the price of this is paid for in length. Unless otherwise specified, our notation and terminology follow Hanna Neumann's book [3].

To prove the first half of the theorem, it is sufficient to find a set of homomorphisms from \( F_c(\mathcal{N}_c) \) to \( F_{c-1}(\mathcal{N}_c) \) whose kernels intersect trivially. Hanna Neumann did just this in the proof of [3, 35.35] for \( c = 4 \), and the same idea works generally: if \( \{ a_1, \ldots, a_c \} \) is a free generating set for \( F_c(\mathcal{N}_c) \) and \( \{ b_1, \ldots, b_{c-1} \} \) is one for \( F_{c-1}(\mathcal{N}_c) \), then the \( 2c - 1 \) homomorphisms \( \delta_i, \ldots, \delta_c, \theta_i, \ldots, \theta_{c-1} \) defined by

\[
\begin{align*}
    a_j \delta_i &= b_j & \text{if } j < i, \\
    a_j \theta_i &= b_j & \text{if } j \leq i, \\
    a_j \delta_i &= 1 & \text{if } j = i, \\
    a_j \theta_i &= b_{j-1} & \text{if } j > i \\
    a_j \delta_i &= b_{j-1} & \text{if } j > i
\end{align*}
\]

will do. The verification of this makes use of the unique representation of the elements of \( F_c(\mathcal{N}_c) \) in terms of basic commutators in \( \{ a_1, \ldots, a_c \} \) as defined by Martin Ward [4]. The case of odd \( c \) is
comparatively straightforward, that of even $c$ requires lengthy and careful discussion.

For the second half, we show that the word $w_e$ defined below is a law in $F_{c-2}(\mathfrak{G}_c)$, and then we exhibit a nilpotent group $G_e$ of class $c$ (and soluble length 3) in which $w_e$ is not a law. The choice of this word was suggested by Higman’s approach. Let $S$ denote the group of all permutations of $\{1, \cdots, c-1\}$, and let $\epsilon(\sigma)$ be 1 or $-1$ according as $\sigma$ is an even or odd permutation; then

$$w_e = \prod_{\sigma \in S} [x_{(c-1)\sigma}, \cdots, x_{1\sigma}]^{\epsilon(\sigma)}.$$ 

If $F = F_{c-2}(\mathfrak{G}_c)$ is freely generated by $\{a_1, \cdots, a_{c-2}\}$, then every element of $F$ can be written in the form

$$d \prod_{j=1}^{c-2} a_j^{u(j)}$$

where $d$ belongs to the commutator subgroup $F'$ of $F$. Substituting $d, \prod_{j=1}^{c-2} a_j^{u(j)}$ for $x_i$ in $w_e$, one finds that every value of $w_e$ in $F$ is of the form

$$\prod_\phi [a_{\phi}, a_{(c-1)\phi}, \cdots, a_{1\phi}]^{\beta(\phi)}$$

where $\phi$ runs through all maps from $\{1, \cdots, c\}$ to $\{1, \cdots, c-2\}$ and

$$\beta(\phi) = \alpha(c, c\phi) \sum_{\sigma \in S} \epsilon(\sigma) \prod_{i=1}^{c-1} \alpha(i\sigma, i\phi).$$

Now two of $1\phi, \cdots, (c-1)\phi$ must be equal: say, $r\phi = s\phi$. Let $T$ be a transversal of the subgroup of $S$ generated by the transposition $(rs)$ such that every element of $S$ not in $T$ can be written uniquely as $(rs)\tau$ with $\tau \subseteq T$. Then

$$\beta(\phi) = \alpha(c, c\phi) \sum_{\tau \subseteq T} \epsilon(\tau) \left\{\prod_{i=1}^{c-1} \alpha(i\tau, i\phi) - \prod_{i=1}^{c-1} \alpha(i(rs)\tau, i\phi)\right\}$$

$$= 0 \text{ because } r\phi = s\phi.$$

Thus $w_e$ is a law in $F$.

The choice of $G_e$ was influenced by the observation that $w_e$ can be rewritten as follows. Put $n = \lfloor \frac{1}{2}(c+1) \rfloor$ so that $c = 2n$ or $2n-1$; note that $n > 1$ as $c > 2$. For each $j$ in $\{1, \cdots, n\}$, let $K_j$ be the subgroup of $S$ generated by the transpositions $(2i-1, 2i)$ with $i \in \{j, \cdots, n-1\}$; thus $K_n$ is the identity subgroup of $S$. Let $T$ denote an arbitrary transversal of $K_1$ in $S$ so that $S = K_1 T$. Then $w_e = w_e^T a_e^T$ where
\[ T^{w_{2n}} = \prod_{\tau \in T} [x_{2n}, x_{(2n-1)r}], [x_{(2n-2)r}, x_{(2n-3)r}], \ldots, [x_{2r}, x_{1r}] \] ^{(r)},

\[ T^{w_{2n-1}} = \prod_{\tau \in T} [x_{2n-1}, [x_{(2n-2)r}, x_{(2n-3)r}], \ldots, [x_{2r}, x_{1r}]]^{(r)}, \]

and \( d_T \) is a law in \( \mathfrak{N}_c \). This is a consequence of the fact that

\[ [y, [x_{2n-2}, x_{2n-1}], \ldots, [x_2, x_1]] \]

\[ = d_j \prod_{\sigma \in K_j} [y, x_{(2n-2)\sigma}, \ldots, x_{(2j-1)\sigma}, [x_{(2j-2)\sigma}, x_{(2j-3)\sigma}], \ldots, [x_{2r}, x_{1r}]]^{(r)} \]

where \( d_j \) is a law in \( \mathfrak{N}_c \); a tautology if \( j = n \) and \( d_n = 1 \), and proved in general by reverse induction on \( j \).

The group \( G_c \) is constructed as follows. Let \( W_n = F_{2n}(\mathfrak{N}_n \setminus \mathfrak{N}^2) \) be freely generated by \( C = \{ v_2, u_4, \ldots, u_{2n}, v_1, \ldots, v_n \} \). For each \( i \) in \( \{1, \ldots, n\} \), there is an automorphism \( \gamma_i \) of \( W_n \) which maps \( u_{2i} \) to \( u_{2i-1} \) and fixes all other elements of \( C \). It is easy to check that \( \gamma_1, \ldots, \gamma_n \) generate a free abelian subgroup of rank \( n \) in the automorphism group of \( W_n \). The group \( H_n \) is the splitting extension of \( W_n \) by a free abelian group of rank \( n \) freely generated by \( u_1, u_2, \ldots, u_{2n-1} \) where the homomorphism specifying the extension maps \( u_{2i-1} \) to \( \gamma_i \) for all \( i \). We take \( G_c \) to be the largest nilpotent-of-class-\( c \) factor group of the subgroup of \( H_n \) generated by \( \{ u_1, \ldots, u_c \} \).

We now sketch the verification that this \( G_c \) has the required properties. If \( z_1, \ldots, z_r \in C \), then \( [z_1, \ldots, z_r, u_{2i-1}] \) is a (possibly empty) product of left-normed commutators of weight at least \( r \) with entries from \( C \) in which all the commutators of weight \( r \) have fewer entries \( u_{2i} \) than \( [z_1, \ldots, z_r] \). From this, a routine argument shows that every left-normed commutator of weight \( 2n \) with entries from \( \{ u_1, \ldots, u_{2n} \} \) lies in the \( n \)th term of the lower central series of the subgroup of \( W_n \) generated by \( \{ v_1, \ldots, v_n \} \). It follows that \( H_n \) has class \( 2n \).

Suppose that \( c = 2n \), so that \( G_c = H_n \), and evaluate \( w_{2n} \) when \( u_i \) is substituted for \( x_i \). Since each \( d^n \) is a law in \( H_n \), we get the same value from substituting in any \( w^n \), and we exploit our freedom in choosing \( T \). Let \( R \) be the group of all permutations on \( \{1, \ldots, n-1\} \); define a monomorphism \( \rho \rightarrow \rho^* \) from \( R \) to \( S \) by \( (2k-1)\rho^* = 2(kp)-1 \), \( (2k)\rho^* = 2(kp) \) if \( k < n \) and \( (2n-1)\rho^* = 2n-1 \), and denote its image by \( R^* \). As \( R^* \) avoids \( K_1 \), it can be extended to a transversal \( T \). Observe that \( [u_i, u_j] \) lies in the commutator subgroup of \( W_n \) unless \( i-j = 1 \) and \( \max(i, j) \) is even; hence
\[
\left[ [ U_{2n}, U_{(2n-1)r} ], [ U_{(2n-2)r}, U_{(2n-3)r} ], \ldots, [ U_{2r}, U_{1r} ] \right] = 1
\]

unless \( r \in K_1R^* \); and \( K_1R^* \cap T = R^* \). Finally, note that every element of \( R^* \) is even. Using this information, we find that

\[
\prod_{\sigma \in S} \left[ U_{2n}, U_{(2n-1)r}, \ldots, U_{1r} \right]^{x(\sigma)}
= \prod_{\rho^* \in R^*} \left[ [ U_{2n}, U_{(2n-1)r}^\rho* ], \ldots, [ U_{2r}, U_{1r}^\rho* ] \right]
= \prod_{\rho \in R} \left[ v_n, v_{(n-1)r}, \ldots, v_{1r} \right]
= \prod_{r=1}^{n-1} \left[ v_n, v_r, v_1, \ldots, v_{r-1}, v_{r+1}, \ldots, v_{n-1} \right]^{(n-2)!},
\]

where the last equality holds because \( W_n \) is metabelian. By [3, 36.32], this value of \( w_{2n} \) is not trivial. This proves that if \( c \) is even, \( w_c \) is not a law in \( G_c \).

A similar argument shows that if \( c = 2n - 1 \) and \( u_i \) is substituted for \( x_i \), the value of \( w_{2n-1} \) is

\[
\prod_{r=1}^{n-1} \left[ U_{2n-1}, v_r, v_1, \ldots, v_{r-1}, v_{r+1}, \ldots, v_{n-1} \right]^{(n-2)!},
\]

which does not belong to the \( 2n \)th term of the lower central series of \( H_n \). Thus also when \( c \) is odd, \( w_c \) is not a law in \( G_c \), and this completes the outline of the proof.

**Remark (added in proof July 9, 1968).** An alternative and independent proof of this result has been obtained by Professor F. Levin and submitted to J. Austral. Math. Soc.

**References**


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