Consider the following diagram of pointed spaces and maps

where \( p_g = f \) and \( p \) is a fibration with fiber \( F \). Suppose that \( X \) is a CW-complex of dimension \( \leq 2\text{conn}(F) \) and \( \text{conn}(F) \leq 1 \) (\( \text{conn} \) = connectivity). Let \([X, Y]_B\) be the set of homotopy classes of pointed maps over \( f(H : X \times I \rightarrow Y) \) is a homotopy over \( f \) if \( pH_t = f \) for each \( t \in I \). Becker proved in [2], [3] that under these hypotheses \([X, Y]_B\) can be given an abelian group structure with \([g]\) as zero element.

The purpose of this note is to describe a spectral sequence of the Adams type which converges to \([X, Y]_B\). The differentials of the spectral sequence are the twisted operations described in [6], [7]. The sequence has the same relation to the method of computing \([X, Y]_B\) used in [6], [7] as the Adams spectral sequence has to the killing-homotopy method of computing ordinary homotopy groups. This note should be read as a sequel to [7].

A different spectral sequence for \([X, Y]_B\) is given by Becker in [3]. A sequence apparently similar to the one to be described here is mentioned in [4] and credited to Becker and Milgram.

1. The spectral sequence. Consider the following commutative diagram:

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where $Y^2$ is the square of $Y$, i.e. the pullback of $p$ by $p$, and $s$ is the canonical cross section. Write $(Y^2, Y)$ for $(Y^2, s(Y))$. Let $A = A_p$ be the mod $p$ Steenrod algebra and use $Z_p$ coefficients for all cohomology. Let $i: F \subset Y^2$ and assume that $i^*: H^*(Y^2) \to H^*(F)$ is onto. Assume also that $H_j(F; Z)$ is finitely generated for each $j$. Let $A(Y) = H^*(Y) \odot A$ be the Massey-Peterson algebra [5]. Then $H^*(Y^2, Y)$ and $H^*(X, *)$ are $A(Y)$ modules via $p: Y^2 \to Y$ and $g: X \to Y$.

Theorem. Under the above hypotheses, there is a spectral sequence such that

1. $E_2^{pq} = \text{Ext}_{H^*(F)}^p(H^*(Y^2, Y), H^*(X, *))$
2. $E_2^{pq} = B^{p,q}/B^{p+1,q+1}$, where $[X, Y]_B = B^0 \supset B^1 \supset B^2 \supset \ldots$

and $\cap B^{p,q} = \text{all elements of } [X, Y]_B$ of finite order prime to $p$.

Notes. (1) $H^*(Y^2, Y)$ can be easily computed as an $A(Y)$ module in terms of $H^*(Y)$ by the results of [5].
(2) Low level computations with the spectral sequence are not difficult. However, the results can be obtained also, and sometimes more easily, by the methods of [6], [7]. The spectral sequence should ultimately prove valuable for proving general theorems about $[X, Y]_B$ (e.g., about immersion groups).
(3) If $B = *$ (a point) then the spectral sequence reduces, after a little manipulation of $E_2$, to the Adams spectral sequence for $[X, Y]$.

2. Sketch of the proof. Let $\mathcal{Y}$ be the category of all triples $(Z, \xi, \delta)$ where $Y \xrightarrow{\xi} Z \xrightarrow{\delta} Y$ and $\xi \delta = 1$, i.e., of all coretractions of $Y$ with given retraction. A morphism in the category is a map $m: Z \to W$ such that $m\xi = \omega$ and $\omega \delta = \eta$. Recall from [6], [7] that one can define a notion of homotopy in $\mathcal{Y}$ (in the obvious way) and also cone, suspension, path, and loop functors enjoying the same properties as the usual functors on $\mathcal{Y}$ (= the ordinary category of pointed spaces and maps). The cone-suspension sequence (Puppe sequence) and the path-loop sequence are exact after application of $(\sim, Z)$ and $(Z, \sim)$ respectively. $(\sim, \sim)$ denotes the set of homotopy classes of maps in the category. In brief, all the notions concerning $\mathcal{Y}$ generalize to $\mathcal{Y}$.

We will now apply an upside down version of Adams' method [1] to $g: Y^2 \to Y$. Since $[X, Y]_B = [X, Y^2] = [X \vee Y, Y^2]$, we can work in $\mathcal{Y}$. Suspension of $Y^2$ in $\mathcal{Y}$ has the effect of suspending $F$ in $\mathcal{Y}$. Successively larger pieces of the spectral sequence are obtained by taking successively higher suspensions of $Y^2$. We will be content here with one piece. Assume $\text{conn}(F) = n$. Consider the following commutative diagram in $\mathcal{Y}$. 

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Each $A_i$ is a product of $K(Z_p, j)$'s, $a_1 = (q, u)$, where $u = (u_1, u_2, \cdots)$ and the $i^*u_j$'s form a set of $A$ generators for $H^j(F)$, $j \leq 2n + 1$. $v_m = (v_{m,1}, v_{m,2}, \cdots)$ and the $v_{m,j}$'s form a set of $A(X)$ generators for $(\ker a_m)^j$, $j = 2n + 1$.

The tower can be formally written as a new tower in $3Y$ simply by replacing $A_m$, $m > 0$, by $Y \times A_m$ and $v_m$ by $(q_m, v_m)$ where $q_m: Y_m \to Y$ is from the original tower. Each fibration $Y_m \to Y_{m-1}$ is a fibration in $3Y$ induced from a principal fibration in $3Y$.

Now apply the functor $(X \vee Y, -)$. The resulting exact couple gives the promised piece of the sequence.

REFERENCES