AN AXIOMATIC APPROACH TO THE BOUNDARY 
THEORIES OF WIENER AND ROYDEN 

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In this note we announce results, obtained in the framework of
Brelot's axiomatic potential theory, which are applicable to the
Wiener and Royden boundary theories for Riemann surfaces.² Recall
that in Brelot's theory, we consider a sheaf 𝒞 of real-valued functions
with open domains contained in a locally compact, noncompact, con­
nected and locally connected Hausdorff space 𝒮, with the functions
satisfying certain axioms. Specifically, by a harmonic class of func­
tions on 𝒮 we mean a class 𝒞 of real-valued continuous functions
with open domains. For each open 𝛪 ⊆ 𝒮, 𝒞₀ denotes the set of func­
tions in 𝒞 with domains equal to 𝛳; it is assumed that 𝒞₀ is a real
vector space. The three axioms of Brelot which 𝒞 is assumed to
satisfy are (1) a function is in 𝒞 if and only if it is locally in 𝒞;
(2) there is a base for the topology of 𝒮 which consists of regions
regular for 𝒞, i.e. connected open sets 𝛲 such that any continuous
function 𝑓 on 𝜀 has a unique continuous extension in 𝒞 which
is nonnegative if 𝑓 is nonnegative; (3) the upper envelope of any in­
creasing sequence of functions in 𝒞₀ where 𝑀 is a region (i.e. open and
connected) is either +∞ or an element of 𝒞₀.

Let 𝒞⁻ and 𝒞₋ denote the classes of functions which are super-
harmonic and subharmonic with respect to 𝒞; let 𝒞₋⁻ denote the
subclass of 𝒞⁻ consisting of functions bounded below. We assume
as another axiom: (4) 1 ∈ 𝒞₋⁻.

Let 𝒮 be a Hausdorff space in which 𝒮 is imbedded as a dense
(and therefore open) subspace, and henceforth let us agree that 𝒮
will mean the closure of 𝒮 in 𝒮 and 𝜃 = 𝒮 − 𝒮. If 𝑀 is an open subset
of 𝒮, we shall say that 𝜃 is associated with 𝒞₋⁻ if every 𝑣 ∈ 𝒞₋⁻ whose
limit inferior is nonnegative at every point of 𝜃 is necessarily
nonnegative on 𝑀. Throughout this note, we shall denote lim sup
by lim sup; similar notation is used for lim inf and lim sup.

**Theorem 1.1. If 𝜃 is an open subset of 𝒮 and 𝜃 is associated with
monic and subharmonic with respect to 𝒞; let 𝒞⁻⁻ denote the
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² These results will appear with proofs as part of a forthcoming article in the
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Assume that $\partial W$ is associated with $\mathcal{E}^B_W$; then given a bounded real-valued function $f$ on $\partial \Omega$ (where $\Omega$ is an open subset of $W$) one can define $H^-(f, \Omega) \subseteq \mathcal{E}$ to be the lower envelope of the set $\{v \in \mathcal{E}_q^B : \lim \inf_v v(x) \geq f(x) \text{ for all } x \in \partial \Omega \}$ and dually define $H^-(f, \Omega)$ as respectively the upper- and lower-$\mathcal{E}$-extensions of $f$ in $\Omega$. If they are equal, we say that $f$ is resolutive on $\partial \Omega$. A point $x_0 \in \partial \Omega$ for which $\limsup H^-(f, \Omega)(x_0) \leq \lim \sup f(x_0)$ for every bounded function $f$ on $\partial \Omega$ is said to be regular (with respect to $\mathcal{E}$). Given $x_0 \in \partial \Omega$, a positive function $b \in \mathcal{E}$ defined in the intersection of $\Omega$ with an open neighborhood of $x_0$ and for which $\lim b(x_0) = 0$ is called an $\mathcal{E}$-barrier (or simply a barrier) for $\Omega$ at $x_0$.

We say that there is a system of barriers for $\Omega$ (or, for emphasis, $\Omega$) at $x_0$ if there is a base $\theta$ for the neighborhood system of $x_0$ such that on the intersection of $\Omega$ with $\omega \cap \partial \Omega$ there is defined a barrier $b$ for $\Omega$ at $x_0$ with

$$\inf \{\lim \inf b(x_i) : x_i \in \partial(\omega \cap \Omega) - (\omega \cap \partial \Omega)\} > 0.$$  

Such a barrier is said to belong to $\Omega$ and $\omega$. An $\mathcal{E}$-unit-barrier for $\Omega$ at $x_0$ is a function $b_1 \in \mathcal{E}$, defined on the intersection of $\Omega$ with a neighborhood of $x_0$ and such that $\lim b_1(x_0) = 1$. With these definitions, we have

THEOREM 1.2. Let $x_0$ be a point of $\partial \Omega$. Assume there is a system of barriers and an $\mathcal{E}$-unit-barrier for $\Omega$ at $x_0$. Then $x_0$ is a regular point for $\Omega$.

2. Let $\mathcal{E}$ be a harmonic class which is hyperbolic on $W$ [5, p. 189], and let $\mathcal{E}_W$ denote the set of all bounded $\mathcal{E}$-harmonic functions on $W$. Then $\mathcal{E}_W$ is a Banach lattice with order unit $H(W)$, where $H(W)$ is the greatest $\mathcal{E}$-harmonic minorant of $1$. The lattice operation $\land_{\mathcal{E}}$ is given by defining $f \land_{\mathcal{E}} g$ to be the least $\mathcal{E}$-harmonic majorant of the pointwise supremum $f \lor g$, and $\land_{\mathcal{E}}$ is similarly defined.

We next consider ideal boundary theory for an arbitrary Banach sublattice $\mathfrak{A}$ of $\mathcal{E}_W$ when $H(W) \in \mathfrak{A}$. Some examples of such sublattices are:

(1) $\mathcal{E}_W$ itself.
(2) $\mathcal{E}_W$ is a Banach lattice with order unit $H(W)$, where $H(W)$ is the greatest $\mathcal{E}$-harmonic minorant of $1$. The lattice operation $\land_{\mathcal{E}}$ is given by defining $f \land_{\mathcal{E}} g$ to be the least $\mathcal{E}$-harmonic majorant of the pointwise supremum $f \lor g$, and $\land_{\mathcal{E}}$ is similarly defined.

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(b) \( D(f,f) + \int_W P^2 < \infty \) where \( D(f,f) \) is the Dirichlet integral of \( f \).

Let a Banach sublattice \( \mathcal{S} \) of \( \mathfrak{B}^\infty \) containing the order unit, \( H(W) \), be given. Now form the \( Q \)-compactification \([2, \text{pp. 96–97}]\) and \([\alpha]\) \( W^*_{\mathcal{S}} \) of \( W \) with \( Q = \mathcal{S} \); this is a compact Hausdorff space containing \( W \) as a dense subspace, determined up to homeomorphism by the properties that each \( f \in \mathcal{S} \) has a continuous extension to \( W^*_{\mathcal{S}} \) and that the family of all these extensions separates the points of \( \Delta_{\mathcal{S}} = W^*_{\mathcal{S}} - W \). Define

\[
\Gamma_{\mathcal{S}} = \{ t \in \Delta_{\mathcal{S}} : H(W)(t) = 1 \} \cap \bigcap_{f, g \in \mathcal{S}} \{ t \in \Delta_{\mathcal{S}} : (f \wedge g)(t) = (f \wedge g)(t) \}
\]

and let \( W_{\mathcal{S}} = W \cup \Gamma_{\mathcal{S}} \). Then

**Theorem 2.1.** \( \Gamma_{\mathcal{S}} \) is associated with \( \mathfrak{K}^b_{W} \), whence \( \Gamma_{\mathcal{S}} \) is nonempty.

**Theorem 2.2.** If \( M \subseteq \Delta_{\mathcal{S}} \) is a closed set which is associated with \( \mathfrak{K}^b_{W} \), then the restriction map \( f \mapsto f|_M \) of \( \mathcal{S} \) into \( \mathfrak{C}_R(M) \) is an isometry (not necessarily onto) preserving positivity in both directions.

Now by the lattice form of the Stone-Weierstrass theorem we have

**Theorem 2.3.** The restriction mapping \( f \mapsto f|_{\Gamma_{\mathcal{S}}} \) of \( \mathcal{S} \) into \( \mathfrak{C}_R(\Gamma_{\mathcal{S}}) \) is a surjective isometry sending the order unit of \( \mathcal{S} \) to the order unit \( 1 \) of \( \mathfrak{C}_R(\Gamma_{\mathcal{S}}) \) and preserving the lattice operations.

**Theorem 2.4.** \( \Gamma_{\mathcal{S}} \) is the intersection of all sets \( \Gamma_p = \{ t \in \Delta_{\mathcal{S}} : \lim \inf_p p(t) = 0 \} \) as \( p \) ranges through the \( \mathfrak{K} \)-potentials on \( W \). No proper closed subset of \( \Gamma_{\mathcal{S}} \) is associated with \( \mathfrak{K}^b_{W} \).

**Theorem 2.5.** Except perhaps when \( \mathcal{S} \) consists only of constant functions, there is an \( \mathfrak{K}_{\infty} \)-unit barrier and a system of barriers for \( W^*_{\mathcal{S}} \) at each point of \( \Gamma_{\mathcal{S}} \), whence each \( x \in \Gamma_{\mathcal{S}} \) is regular with respect to any open set \( \Omega \subseteq W \) for which \( x \in \partial \Omega \cap \Gamma_{\mathcal{S}} \). (Here \( \partial \Omega \) is taken in \( W^*_{\mathcal{S}} \).)

**Theorem 2.6.** Let \( \mathfrak{S} \) denote those bounded functions in \( \mathfrak{K}^b_{W} \) for which the greatest \( \mathfrak{K} \)-harmonic minorant is in \( \mathcal{S} \). For any \( v \in \mathfrak{S} \), let \( I(v) \) be the function on \( \Gamma_{\mathcal{S}} \) defined by \( I(v)(t) = \lim \inf_W v(t) \) for each \( t \in \Gamma_{\mathcal{S}} \). Then \( I(v) \) is continuous on \( \Gamma_{\mathcal{S}} \) for each \( v \in \mathfrak{S} \), and the mapping \( I : \mathfrak{S} \mapsto \mathfrak{C}_R(\Gamma_{\mathcal{S}}) \) is positively homogeneous and additive.

If \( W \) is an open Riemann surface, \( \mathfrak{K} \) the class of harmonic functions in the usual sense, and \( \mathfrak{S} = \mathfrak{B}^\infty \), then \( \Gamma_{\mathcal{S}} \) is homeomorphic to the harmonic part of the Wiener boundary even though \( \Delta_{\mathcal{S}} \) is "smaller" than the Wiener boundary. If \( \mathfrak{S} \) is the uniform closure of \( \mathfrak{B}^\infty \), the bounded harmonic functions with finite Dirichlet integrals, then \( \Gamma_{\mathcal{S}} \)
is the harmonic part of the Royden boundary and $\mathcal{B}_{\Omega}^{\mathcal{W}}$ is isometrically isomorphic to a dense subset of $\mathcal{C}_{\mathcal{R}}(\Gamma_{\Phi})$.

**BIBLIOGRAPHY**


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