SCHAUDER DECOMPOSITIONS IN BANACH SPACES

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A sequence \( (M_i) \) of closed subspaces of a Banach space \( E \) is called a Schauder decomposition of \( [M_i] \), the smallest subspace containing \( \cup M_i \), if every element \( u \) of \( [M_i] \) has a unique, norm convergent expansion \( u = \sum u_i \), where \( u_i \in M_i \) for \( i = 1, 2, \cdots \). It is well known (see, e.g. [2]) that any sequence \( (u_i) \subseteq E \) with \( 0 \neq u_i \in M_i \) for \( i = 1, 2, \cdots \) is basic (i.e., a basis for its closed linear span). The converse of this statement is not true, but we do derive the following theorem, and mention several corollaries.

**Theorem.** Let \( (M_i) \) be a sequence of closed subspaces of the Banach space \( E \) such that each sequence \( (u_i) \subseteq E \) with \( 0 \neq u_i \in M_i \) is basic. Then there exists an integer \( N \) such that \( (M_i)_{i \geq N} \) is a Schauder decomposition of \( [M_i]_{i \geq N} \).

To simplify the proof of the theorem, we use the following characterization of Schauder decompositions due to Grinblyum [3]. A sequence \( (M_i) \) of closed subspaces of \( E \) is a Schauder decomposition of \( [M_i] \) if and only if there exists a constant \( K \) such that for all integers \( n, m \) and all sequence \( (u_i) \) with \( u_i \in M_i \), \( \sum_{i=1}^{n} u_i \leq K \sum_{i=1}^{m} u_i \).

We note that this norm condition may be replaced by \( \sum_{i=1}^{n} a_i u_i \leq K \sum_{i=1}^{m} a_i u_i \) where the scalars \( (a_i) \) are also arbitrary. Since a sequence \( U = (u_i) \) is basic if and only if there exists \( K = K(U) \) such that this last inequality holds for all \( (a_i), m \) and \( n \), we see that each \( (u_i) \) with \( u_i \in M_i \) is basic if \( (M_i) \) is a Schauder decomposition.

Let \( U = (u_i) \) be a sequence with \( 0 \neq u_i \in M_i \), and set \( U_n = (u_i)_{i \geq n} \). Let \( K(U_n) \) be the smallest constant such that \( \| \sum_{i=-n}^{p} a_i u_i \| \leq K \| \sum_{i=-n}^{p} a_i u_i \| \) holds for all \( K \geq K(U_n) \), all \( (a_i) \) and integers \( p, q \).

**Lemma.** Let \( (M_i) \) be a sequence of closed subspaces of \( E \) such that each \( U = (u_i) \) with \( 0 \neq u_i \in M_i \) is basic. Then there exists an integer \( N \) and a constant \( K \geq 1 \) such that every sequence \( U \) as above has \( K(U_n) \leq K \).

**Proof.** If \( K \) and \( N \) do not exist, then for each integer \( n \) and each \( M \geq 1 \), there exists a \( U \) with \( K(U_n) > M \) (noting \( K(U_{n+1}) \leq K(U_n) \)). Choose \( U^{(1)} \) so that \( K(U^{(1)}) > 2 \). Then there exist integers \( q_1 > p_1 \) such that \( \| \sum_{i=1}^{p_1} a_i u_i^{(1)} \| > 2 \| \sum_{i=1}^{p_1} a_i u_i^{(1)} \| \) for some sequence \( (a_i) \). Similarly, there exist \( U^{(2)} \) and \( q_2 > p_2 \) such that

\[\sum_{i=1}^{p_2} a_i u_i^{(2)} \]

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and in general we get $U^{(j)}$ and integers $p_j$, $q_j$ such that $q_{j-1} < p_j < q_j$ and

$$\left\| \sum_{i=q_{j-1}+1}^{p_i} a_i u_i^{(j)} \right\| > 2^j \left\| \sum_{i=q_{j-1}+1}^{q_i} a_i u_i^{(j)} \right\|.$$ 

With these bounds, the sequence $U$ defined by $u_i = u_i^{(j)}$ if $q_{j-1} < i \leq q_j$ is not basic, which is a contradiction proving the lemma.

The theorem follows immediately from the lemma and the Grinblyum criterion.

To see that $N$ is in general greater than 1, let $E$ be separable, $(x_i)$ a basic sequence in $E$ such that $\text{codim } [x_i] = \infty$ and $E_i$ a closed subspace of $E$ which is a quasicomplement but not a complement of $[x_i]$ in $E$. (For a construction of such an $E_i$ see Gurarii and Kadec [4].) If we set $M_1 = E_1$, $M_2 = [x_1]$, $M_3 = [x_2]$, etc., each sequence with just one element in each $M_i$ is basic, but $(M_i)$ is not a Schauder decomposition of $E$ since $M_1 + [x_i] \subsetneq E$. In order to have $N = 1$, then, we must keep $[M_i] i < N$ from being a quasicomplement of $[M_i] i \geq N$ for each $N$. In fact, the addition of this hypothesis is also sufficient, for then we see that $[M_i] = M_1 \oplus M_2 \oplus \cdots \oplus M_{N-1} \oplus [M_i] i \geq N$, and so $(M_i)$ is a Schauder decomposition of $[M_i]$. These corollaries are now immediate. In each, we let $U$ be an arbitrary sequence $(u_i)$ with $0 \neq u_i \in M_i$, and call $U$ a proper sequence.

**Corollary.** A sequence $(M_i)$ of closed subspaces of $E$ is a Schauder decomposition if and only if (a) $[M_i] = [M_i] i < n \oplus [M_i] i \leq n$ and (b) each proper sequence $U$ is basic.

**Corollary.** The previous corollary holds with (a) replaced by (a') 

$$[M_i] = M_k \oplus [M_i] i = k$$

for each $k$.

**Corollary.** Let $(M_i)$ be a sequence of finite-dimensional subspaces of $E$. Then $(M_i)$ is a Schauder decomposition if and only if each proper sequence is basic.

It is easy to see that an $N$ dimensional Banach space $F$ has a basis $(f_i)_{i=1}^{N}$ such that

$$\left\| \sum_{i=1}^{p} a_i f_i \right\| \leq N \left\| \sum_{i=1}^{N} a_i f_i \right\|.$$
always holds (using, for example, the result of Taylor [5]). The author does not know what the best bound that can replace \( N \) in general will be, but it must be greater than 1 (see, e.g. [1]). However, using the last corollary, and the \( N \)-bound above, we obtain:

**Proposition.** Let \( \dim M_i \leq N_i \), \( M_i \subseteq E \) for \( i = 1, 2, \cdots, \) and \( [M_i] = E \). Set \( N_j = \dim M_1 + \dim M_2 + \cdots + \dim M_j \). The following are equivalent

(a) \((M_i)\) is a Schauder decomposition of \( E \),
(b) \( E \) has a basis \((x_i)\) with \( M_j = [x_i \mid N_{j-1} < i \leq N_j] \),
(c) each proper sequence is basic.

The proof of the proposition is a routine exercise.

**Problem.** Does the previous result hold with the weaker assumption \( \dim M_i < \infty \)?

**References**

1. F. Bohnenblust, *Subspaces of \( l_{m,n} \) spaces*, Amer. J. Math. 63 (1941), 64–72.

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