We present here in general terms the idea of the mean of a function relative to a "weight function" $w(\xi, \nu)$, special instances and applications appearing elsewhere [1], [2].

1. **The weight function.** If $X = [h, k]$ is a real interval, $(I, A, \mu)$ a finite measure space with $\mu(I) = 1$, and $w(\xi, \nu)$ a nonnegative function on $X \times I$ which, for each $\nu$ of $I$, is measurable, and positive a.e. on $X$, then the indefinite integral

$$W(x, \nu) = \int_h^x w(\xi, \nu) d\xi$$

is defined on $X \times I$, and the function

$$\mathcal{W}(x) = \int_I W(x, \nu) d\mu, \quad x \in X$$

which we assume to exist, is continuous and strictly increasing on $X$, as is $W(x, \nu)$ for each $\nu$.

2. **The mean of a function.** Let $x(\nu)$ be any $\mu$-integrable function on $I$ to $X$ for which the integral functional

$$\mathcal{W}_x = \int_I W(x(\nu), \nu) d\mu$$

exists. Let $x_u$ be the essential upper bound of $x(\nu)$ on $I$, i.e., the g.l.b. of all real $x$ for which $\mu \{ \nu \mid x(\nu) > x \} = 0$, the essential lower bound

---

1 Work performed under the auspices of the U. S. Atomic Energy Commission.
and $x_i$ being analogously defined. Clearly $x(\nu)$ is constant $\mu$-a.e. if and only if $x_i = x_u$.

Referring to (1), it is apparent that the continuous, strictly increasing function

$$B(x) = \int_I \int_{z(\nu)}^z w(\xi, \nu) d\xi d\mu = \mathcal{W}(x) - \mathcal{W}_z$$

has a unique zero $b$ on $X$, namely

$$b = \mathcal{W}^{-1}(\mathcal{W}_z)$$

called the mean of $x(\nu)$ relative to $w(\xi, \nu)$. For, if $x(\nu)$ is a constant $x_0$, $\mu$-a.e., we have $B(x_0) = 0$; otherwise we see that $B(x_i) < 0 < B(x_u)$, so that $B(b) = 0$ for a unique $b$ on $(x_i, x_u)$.

3. The principal theorem. For an arbitrary bounded, monotone nondecreasing function $g(\xi)$ on $X$, we analogously define

$$G(x, \nu) = \int_I \int_{z(\nu)}^z g(\xi) w(\xi, \nu) d\xi$$

on $X \times I$, and assume the existence of

$$\mathcal{G}(x) = \int_I G(x, \nu) d\mu, \quad x \in X$$

and of

$$\mathcal{G}_z = \int_I G(x(\nu), \nu) d\mu.$$ 

For the function

$$C(x) = \int_I \int_{z(\nu)}^z g(\xi) w(\xi, \nu) d\xi d\mu = \mathcal{G}(x) - \mathcal{G}_z$$

we then have the basic

**Theorem.**

(2) \[ C(b) \leq 0 \]

or

$$\mathcal{G}(\mathcal{W}^{-1}(\mathcal{W}_z)) \leq \mathcal{G}_z.$$ 

Equality holds if and only if $x(\nu) \equiv b$, $\mu$-a.e., or $g(\xi) \equiv g(b)$ everywhere on the open interval $(x_i, x_u)$.
The inequality is rendered transparent by splitting $I$ into the $\mu$-measurable subsets $L, Z, U$ on which $x(\nu) \leq b$, respectively, and observing that

$$-C(b) = g(b)B(b) - C(b)$$

$$= \int_L \int_{x(\nu)}^b \{g(b) - g(\xi)\} w(\xi, \nu) d\xi d\mu$$

$$+ \int_U \int_b^{x(\nu)} \{g(\xi) - g(b)\} w(\xi, \nu) d\xi d\mu \geq 0.$$ 

4. Two applications. In the simplest case, $w(\xi, \nu) = 1$, (2) is Jensen's inequality

$$G\left(\int_I x(\nu) d\mu\right) \leq \int_I G(x(\nu)) d\mu$$

for the general convex function $G(x) = \frac{G(x)}{G(x, \nu)} = \int_{\nu}^b g(x) d\xi$ [3, §13.34, §18.43]. A particular instance is mentioned in [2, §3].

Again, if we take $h > 0$ and set $w(\xi, \nu) = \xi^{-s}$, $s$ real, we find that $b$ is the "mean of order $s$" of $x(\nu)$:

$$M_s = \left\{\left(\int_I x(t) d\mu\right)^{1/s}, s \neq 0; \exp\int_I \log x(\nu) d\mu, s = 0\right\},$$

and (2) yields the classical inequality

$$M_s \leq M_t$$

if one takes $g(\xi) = \xi^{s-t}$ [1], [2].

These are trivial examples of the "separable" case $w(\xi, \nu) = w(\xi)f(\nu)$. Nonseparable cases arise naturally in physical problems, as indicated below.

5. A "minimax" principle. Let $y(\nu)$ be a second function such as $x(\nu)$, and suppose that

$$\int_I \int_{x(\nu)} y(\nu) w(\xi, \nu) d\xi d\mu = 0.$$ 

This is equivalent to the assertion that $y(\nu)$ and $x(\nu)$ have the same mean $b$ relative to $w(\xi, \nu)$, and it follows at once from the Theorem (applied to $y(\nu)$) that
If we regard \( x(\nu) \) as initial temperature distribution on an interval \( I \), of mass \( m(\nu) \) per unit length \( (d\mu = m(\nu) d\nu) \), and specific heat \( w(\xi, \nu) \), then (3) singles out the energy conserving distributions \( y(\nu) \), and (4) (with \( g(\xi) = -1/\xi \)) shows that, among these, the entropy change is greatest for the uniform mean temperature \( y(\nu) \equiv b \).

REFERENCES


2. ———, *The mean of a function \( x(\nu) \) relative to a function \( w(\xi, \nu) \)*, Amer. Math. Monthly (to appear).


*University of California, Los Alamos Scientific Laboratory*