ON GAUSSIAN SUMS

BY TAKASHI ONO

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This note is an outline of some of the author's recent work on a generalization of Fourier transforms in adele spaces. Here we treat only the simplest case. The details and a generalization for an arbitrary ground field and a system of polynomials will be given elsewhere. For the unexplained notions, see [1], [2] and [3].

Let \( f(X) \) be an absolutely irreducible polynomial in \( \mathbb{Q}[X] = \mathbb{Q}[X_1, \cdots, X_n] \) such that the corresponding hypersurface \( H = \{ x \in \mathbb{Q}^n ; f(x) = 0 \} \) is nonsingular, where \( \Omega \) denotes a universal domain containing \( \mathbb{Q} \). Let \( V \) be the complement of \( H \) in \( \Omega^n \) viewed as an algebraic variety in \( \Omega^{n+1} \) in an obvious way. Hence the \( n \)-form \( \omega = f^{-1}dx = dx_1 \wedge \cdots \wedge dx_n \), is everywhere holomorphic and never zero on \( V \). For each valuation \( v \) of \( \mathbb{Q} \), denote by \( \mathbb{Q}_v \) the completion of \( \mathbb{Q} \) at \( v \). Denote by \( \mathcal{A}, \mathcal{A}^* \) the adele ring and the idele group of \( \mathbb{Q} \), respectively. For an idele \( a \in \mathcal{A}^* \), \( |a|_A \) will denote the module of \( a \). The adelization \( \mathcal{V}_A \) of \( V \) is then given by \( \mathcal{V}_A = \{ x \in \mathcal{A}^n ; f(x) \in \mathcal{A}^* \} \). We denote by \( \mathcal{S}(\mathcal{Q}_v^*) \), \( \mathcal{S}(\mathcal{A}_v^n) \) the space of Schwartz functions on \( \mathcal{Q}_v^*, \mathcal{A}_v^n \), respectively. For each \( v \), the \( n \)-form \( \omega \) on \( V \) induces a measure \( \omega_v \) on \( \mathcal{V}_{Q_v} \) and we know that there is a well-defined measure \( d\mathcal{V}_A \) on \( \mathcal{V}_A \) of the form \( \prod_v \lambda_v^{-1} \omega_v \) with \( \lambda_v = 1 \) and \( \lambda_v = 1 - p^{-1} \). We know that the function

\[
Z(f, \phi, s) = \int_{\mathcal{V}_A} \phi(x) \left| f(x) \right|^s d\mathcal{V}_A, \quad \phi \in \mathcal{S}(\mathcal{A}_v^n),
\]

represents a meromorphic function for \( \text{Re } s > \frac{1}{2} \) having the single simple pole at \( s = 1 \) with the residue \( \int_{\mathcal{A}_v^n} \phi(x) d\mathcal{A}_v^n \), where \( d\mathcal{A}_v^n \) is the canonical measure on \( \mathcal{A}_v^n \) (cf. [4]).

Let \( \chi \) be a basic character of \( \mathcal{A} \) which identifies the additive group \( \mathcal{A} \) with its own dual and let \( \chi \) be the similar character of the additive group \( \mathcal{Q}_v \) induced by \( \chi \). For each \( \xi \in \mathcal{A} \) and \( \phi \in \mathcal{S}(\mathcal{A}_v^n) \), the function \( \phi_\xi(x) = \phi(x)\chi(f(x)\xi) \) is again in \( \mathcal{S}(\mathcal{A}_v^n) \) and hence we have

\[
\text{Res } Z(f, \phi_\xi, s) \overset{\text{def.}}{=} \int_{\mathcal{A}_v^n} \phi(x)\chi(f(x)\xi) d\mathcal{A}_v^n = \mathcal{G}_\xi \phi(\xi).
\]

The transform \( \phi \rightarrow \mathcal{G}_\xi \phi \) is a linear map of \( \mathcal{S}(\mathcal{A}_v^n) \) into the space of con-
continuous functions on $A$, which boils down to the Fourier transform $\phi \mapsto \mathcal{F}\phi$ when $n=1$ and $f(X) = X$.

Now, put $\eta = \text{Res} \, \omega$, the residue form of $\omega$, this being an $(n-1)$-form on the hypersurface $H$ everywhere holomorphic and never zero. When the formal product $\prod_\tau \eta_\tau$ (with no convergence factors) really defines a measure on $H_A$, we say that the canonical measure $dH_A = \prod_\tau \eta_\tau$ exists on $H_A$. The classical theory of trigonometric sums suggests, at least formally, the equality

$$\int_A \mathcal{G}_\tau \phi dA = \int_{H_A} \phi dH \quad \text{for all} \quad \phi \in \mathcal{S}(A^n),$$

where the right-hand side is essentially the singular series for $f(X)$ including the gamma factor. In view of (1), (2), one can interpret (3) as an equality connecting integrals on $V_A$ and $H_A$. More precisely, one can prove the following

**Theorem 1.** If $f(X)$ satisfies the condition

$$(C) \quad \mathcal{G}_\tau \phi \in L^1(A) \quad \text{for all} \quad \phi \in \mathcal{S}(A^n),$$

then the canonical measure $dH_A$ exists, $\phi \big|_{H_A} \in L^1(H_A)$ for all $\phi \in \mathcal{S}(A^n)$ and (3) holds.

Thus, the real problem is to find conditions on $f(X)$ so that (C) holds. For example, (C) is false for $f(X) = X_1 + \cdots + X_n$ with $n \leq 4$. Although we are still far from the complete solution of the problem, we can give the following sufficient conditions.

**Theorem 2.** The condition (C) holds if $f(X)$ satisfies the following two conditions:

(I) $\text{grad} \, f(x) \neq 0$ for all $x \in \Omega^n$,

(II) $\left| \sum_{i \in \mathbb{F}_p} \xi_p \cdot \mathcal{G}_\tau^{(i)} \right| \leq c p^{n-2-\epsilon}$ for almost all $p$, where $c$, $\epsilon$ are positive constants independent of $p$, $f^{(p)}(X)$ is the polynomial over the finite prime field $\mathbb{F}_p$ obtained by reducing the coefficients of $f(X)$, for almost all $p$, and $\xi_p$ is any one of the primitive $p$th roots of 1.

For example, $f(X) = X_1^r X_2^s + X_1 + \sum_{i=3}^n X_i^r$, $r_j \geq 2$, $1 \leq j \leq n$, satisfies (I), (II) whenever $n \geq 7$.

**Remark 1.** The condition (I) implies that $\text{grad} \, f(x) \neq 0$ for all $x \in \mathcal{G}_\tau^n$, for all $\tau$. For this case one proves the stronger result:

1 Such a transform has been introduced by Weil in more general setting [3, Chapter 1, No. 1]. For $p$-adic case, the evaluation of the transform is substantially the Gaussian sum for the polynomial $f(X)$. 
\[ g_P(\xi) = \int_{\mathcal{Q}^*} \phi(x) x_0 (f(x) \xi) dx_0 \in \mathcal{S}(\mathcal{Q}^*) \text{ for all } \phi \in \mathcal{S}(\mathcal{Q}^*). \]

This fact for \( v = \infty \) has been suggested to us by Hörmander. We then found that the same is true for \( v = p \).

**Remark 2.** Unfortunately, the diagonal polynomial \( f(X) = \sum a_i X_i^2 \) does not, in general, satisfy (I). A direct verification of the condition (C) for such a polynomial seems to be not easy because arbitrary Schwartz functions are involved. However, (I) is intrinsic and this might be the case which must precede any attempt at a general theory.

**References**


*University of Pennsylvania, Philadelphia, Pennsylvania 19104*