PERIODIC ORBITS OF HYPERBOLIC DIFFEOMORPHISMS AND FLOWS

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Artin and Mazur in [1] proved that a dense subset of the $C^r$-endomorphisms of a compact differentiable manifold satisfy an exponential growth condition on their isolated periodic points, and they defined a $\zeta$-function which for these endomorphisms has a positive radius of convergence. In [2] and [3] K. Meyer gave a simple proof that hyperbolic diffeomorphisms and flows of Smale [4] which are $C^2$ have exponential growth. It is the purpose of this note to give an even simpler proof of Meyer's theorems in a $C^1$ setting. Since the hyperbolic diffeomorphisms and flows are not dense [5] these results are a long way from including the results of [1].

Let $M$ be a compact differentiable manifold; let $f \in \text{Diff}(M)$ be a $C^1$ diffeomorphism, and let $N_m(f)$ be the number of periodic points of $f$ of period $m$.

**Theorem 1.** Let $f$ satisfy Axiom A of [4, I.6], then there exist constants $c$ and $k$ such that $N_m(f) \leq ck^m$.

**Proof.** $f$ is expansive [4, I.8.7], i.e., $\exists \varepsilon > 0$ such that given $x$, $y$ distinct periodic points of $f \exists n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \varepsilon$. Since $f$ is $C^1$ it is Lipschitz. Let its Lipschitz constant be $k$ which we may choose $> 1$. If $x$ and $y$ are both of period $p$ we may choose $n$ in $0 \leq n < p$ and have $d(x, y) \geq \varepsilon/k^p-1$ by expansiveness. Thus there exists a constant $c$ such that $N_p(f) \leq c V(M) (2k^p-1/\varepsilon)^{\dim M}$ where $V(M)$ is the volume of $M$.

Let $\Phi = \{\phi_t\}$ be a one parameter group acting on $M$, arising from a $C^1$ vector field $X$. Let $N_r(\Phi)$ be the number of closed orbits of $\Phi$ of period less than or equal to $r$.

**Theorem 2.** Let $\Phi$ satisfy Axiom A' of [4, 5.1], then there exist constants $c$ and $k$ such that $N_r(\Phi) \leq ce^{kr}$.

Since the closed orbits are uniformly bounded away from the singularities, which are finite in number, $\Omega_c$ the complement of the singularities in $\Omega$, is compact. Every point $z$ in $\Omega_c$ has a flow box

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neighborhood, $V_z$, such that $V_z$ has a cross section $X_z$; and for some $\delta$, $V_z = \bigcup_{x \in X_z, |t| < \delta} \phi_t(x)$ where $\delta$ is independent of $z$. Denote by $\pi_z: V_z \to X_z$ the map which takes $\phi_t(x)$ to $x$ for $|t| < \delta$ and $x \in X_z$. The $V_z$ and $t$ may be chosen so that if $x, y \in X_z$ and $\phi_t(x), \phi_t(y) \in X_{w_1}$; $\phi_{-t}(x), \phi_{-t}(y) \in X_{w_2}$ then either $\pi_{w_1}\phi_t$ or $\pi_{w_2}\phi_{-t}$ increases their distance by a factor of $k_1 > 1$; $k_1$ independent of $z$.

Now cover $\Omega_0$ by a finite number of $V_z; V_1, V_2, \ldots, V_n$. Note that if $x \in X_z$ and $\phi_t(x) \in V_j$ then $\pi_{w_j}\phi_{-t}\phi_t(x) = x$, and similarly for $\phi_{-t}$. Define an invariant sequence of a closed orbit $\alpha \subset \Omega_0$ as a sequence $a_1 \cdots a_m$ such that $a_i = 1, \ldots, n$; $a_1 = a_m$ and there exist $x_{a_i} \in X_{a_i}$ where $x_{a_1} = x_{a_m}, \pi_{a_1}\phi_t(x_{a_{i-1}}) = x_{a_i}$ and $\pi_{a_{i-1}}\phi_{-t}(x_{a_i}) = x_{a_{i-1}}$. It is clear from the definition and the choice of the $V_i$ that no two distinct closed orbits may have the same invariant sequence. We will show that $\exists c>0$ such that a closed orbit of period $\leq \tau$ has an invariant sequence of length at most $n\tau r + 1$, and thus the number of closed orbits of period $\leq \tau$ is less than or equal to $n^{n\tau r + 2} = n^{2\tau r \log n}$.

To get the invariant sequence, $\exists c>0$ such that if $\alpha$ is a closed orbit of period $\leq \tau$ then $\alpha$ intersects $X_i$ for any $i$ in at most $cr$ points, $c$ is independent of $\tau$. So there are at most $n\tau r$ intersections of $\alpha$ with all the $X_i$. Let $x_0$ be one of these. Define $x_i$ inductively by $x_{i+1} = \pi_j\phi_t(x_i)$ where $\phi_t(x_i) \in V_j$ for some $j$. $x_k$ must equal $x_m$ for some $k, m \leq n\tau r$ and $k \neq m$.

Of course, this proof would also work in the diffeomorphism case. The construction of the corresponding $V_i$, however, essentially gives the proof of expansiveness which was shown to me by Smale.

These theorems, of course, have relevance for the convergence of the zeta functions. For a discussion of this see [3] and [4].

**References**


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