1. Statement of results. Let $E$ be the total space of a $k$-sphere bundle over the $n$-sphere with characteristic class $\alpha \in \pi_{n-1}(SO_{k+1})$. We consider the problem of classifying, under the relation of orientation preserving diffeomorphism, all differential structures on $E$. It is assumed that $E$ is simply connected, of dimension greater than five, and its characteristic class $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_k)$ (that is, the bundle has a cross-section). In [1] and [2] we gave a complete classification in the special case where $\alpha = 0$. The more general classification Theorems 1 and 2 below include this special case. The proofs of these theorems are sketched in §2 below; detailed proofs will appear elsewhere. J. Munkres [6] has announced a classification up to concordance of differential structures in the case where the bundle has at least two cross-sections. (It is well known that concordance and diffeomorphism are not equivalent, concordance of differential structures being strictly stronger than diffeomorphism.)

**Theorem 1.** Let $E$ be the total space of a $k$-sphere bundle over the $n$-sphere whose characteristic class $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_k)$. Suppose that $2 \leq k < n - 1$. Then, under the relation of orientation preserving diffeomorphism, the diffeomorphism classes of manifolds homeomorphic to $E$ are in a one-to-one correspondence with the equivalence classes on the set $(\theta_n/\theta_{n+1}) \times \theta_{n+k}$, where $(A^*_\alpha, \overline{U}^{n+k})$ and $(B^*_\alpha, \overline{V}^{n+k})$ are equivalent if and only if $A^*_\alpha = \pm B^*_\alpha$ and there exists $\beta \in \pi_k(SO_{n-1})$ such that $U^{n+k} - V^{n+k} = \tau_{n,k}(A^*_\alpha \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta)$.

Theorem 1 is also true in the case where $k = n - 1$ and $n$ is odd. The classification in the case where $n - 1 \leq k \leq n + 2$ is essentially the same as the above and is given in Theorem 2 below. Now we establish the notation used in Theorem 1.

**Notation.** Manifolds and diffeomorphisms are of class $C^\infty$. The group of homotopy $n$-spheres under the connected sum operation $+$

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1 The preparation of this paper was supported in part by National Science Foundation Grant # GP 7036.

2 *Added in proof.* Assume here and in Proposition 2 that $\alpha$ is of order 2 in $\pi_{n-1}(SO_{k+1})$ in the case where $k < n - 3$. This assumption is not made elsewhere.
is denoted by \( \theta_n \), and \( \Phi_n^{k+1} \) is the subgroup of \( \theta_n \) consisting of those homotopy \( n \)-spheres that embed in \((n+k+1)\)-space with a trivial normal bundle. The class of a homotopy \( n \)-sphere \( A^n \) in the group \( \theta_n/\Phi_n^{k+1} \) is denoted by \( A^n_\Phi \). Now let

\[
\sigma_{n-1,k} \colon \pi_{n-1}(SO_k) \otimes \pi_k(SO_{n-1}) \to \theta_{n+k}, \\
\tau_{n,k} \colon \theta_n \otimes \pi_k(SO_{n-1}) \to \theta_{n+k}
\]

be the pairings defined in [4, p. 583]. It is known that these pairings correspond to composition in the stable homotopy groups of spheres. Moreover, it was shown in [2] that \( \tau_{n,k}(\Phi_n^{k+1} \otimes \pi_k(SO_{n-1})) = 0 \), provided that \( k \geq 2 \), and hence the pairing \( \tau_{n,k} \) induces a pairing

\[
\tau'_{n,k} \colon (\theta_n/\Phi_n^{k+1}) \otimes \pi_k(SO_{n-1}) \to \theta_{n+k} \quad (k \geq 2).
\]

**Remark.** If \( k \geq n-3 \), then \( \Phi_n^{k+1} = \theta_n \) (see [1, Lemma 1]) and hence \( \tau_{n,k} = \tau'_{n,k} = 0 \) for \( k \geq n-3 \).

In order to state the result in the case where \( n-1 \leq k \leq n+2 \) we define a function

\[
\sigma_{n,k}' \colon \pi_{n-1}(SO_k) \times \pi_k(SO_n) \to \theta_{n+k}
\]

that is linear in the second variable. The definition of \( \sigma_{n,k}' \) is similar to the definition of the pairing \( \sigma_{n-1,k} \) and is described in §2 below. Now if \( \alpha \in \pi_{n-1}(SO_k) \) is the characteristic class of the bundle \( E \), then we define a homomorphism

\[
\chi_\alpha \colon \pi_k(SO_n) \to \theta_{n+k}
\]

by writing, for each \( \beta \in \pi_k(SO_n) \),

\[
\chi_\alpha(\beta) = \sigma_{n,k}'(\alpha, \beta).
\]

**Theorem 2.** Suppose that the characteristic class \( \alpha \) of the bundle \( E \) may be pulled back to lie in \( \pi_{n-1}(SO_k) \). Then, if \( 1 \leq n-3 \leq k \leq n+2 \) and \( k \geq 2 \), then the diffeomorphism classes of manifolds homeomorphic to \( E \) are in a one-to-one correspondence with the group \( \theta_{n+k}/\text{Image } \chi_\alpha \).

2. **Outline of proofs.** We give \( E \) the “standard” differential structure by making it a smooth \( k \)-sphere bundle over the standard \( n \)-sphere \( S^n \). It is well known that if a \( k \)-sphere bundle over the \( n \)-sphere has a cross-section, then the total space of the bundle has the homology of the product \( S^n \times S^k \). The proof of Theorem 1 is divided into the following four propositions. We use the notation \( E(A^n) \) to denote the differential \((n+k)\)-manifold obtained by making \( E \) into a smooth \( k \)-sphere bundle over a homotopy \( n \)-sphere \( A^n \) in the obvious way (if
$n=4$, then take $A^4$ to be homeomorphic to $S^4$. We assume that $E$ has a cross-section, $n>3$, and $n+k>5$. We also assume that $k \geq 2$, except in Proposition 4 where we allow $k=1$.

**Proposition 1.** If $M$ is a differential $(n+k)$-manifold that is homeomorphic to $E$, then there are homotopy spheres $A^n$ and $U^{n+k}$ such that $M$ is diffeomorphic to $E(A^n) + U^{n+k}$, provided that $k \leq n+2$.

**Sketch of Proof.** Since $E$ is of dimension greater than five and simply connected we can apply the Hauptvermutung of [7] to conclude that there is a PL-homeomorphism $h: M \to E$, where the combinatorial structures are compatible with the differential structures. We try to smooth $h$ by applying the obstruction theory of Munkres [5]. If $k<n$, then the first obstruction to deforming $h$ into a diffeomorphism is an element $c(h)$ in $H_n(M; \Gamma_k)$, where $\Gamma_k$ is the group of diffeomorphisms of $S^{n-1}$ modulo those that extend to diffeomorphisms of the $k$-disk $D^k$. Since $H_n(M; \Gamma_k)$ is isomorphic to $\Gamma_k$ we can consider $c(h)$ to be an element of $\Gamma_k$. Now we construct a manifold $M(c(h))$ and a PL-homeomorphism $j$ from $E$ to $M(c(h))$ such that the first obstruction to smoothing $j$ is $-c(h)$. It follows that the first obstruction to smoothing the composition $jh$ is zero and hence we can suppose that $jh$ is a diffeomorphism modulo the $k$-skeleton. The next step is to show that there is a diffeomorphism modulo a point $\phi: M(c(h)) \to E$, (this is true for $k \leq n+2$) and hence the composition $h' = \phi j h$ is a diffeomorphism modulo the $k$-skeleton. The first obstruction to smoothing $h'$: $M \to E$ is an element $c(h')$ in $H_k(M; \Gamma_n) \cong \Gamma_n$. Now let $A^n$ be the homotopy $n$-sphere that corresponds to $c(h')$ under the isomorphism $\Gamma_n \cong \Theta_n$ ($n \neq 3$). There is a PL-homeomorphism $j'$ from $E$ to $E(A^n)$. Moreover, the first obstruction to smoothing $j'$ is $-c(h')$ and hence we can assume that the composition $j' h'$ is a diffeomorphism up to a point. It follows that there is a homotopy $(n+k)$-sphere $U^{n+k}$ such that $M$ is diffeomorphic to $E(A^n) + U^{n+k}$. The argument in the case where $n \leq k \leq n+2$ is essentially the same. Note that if $n=4$, then the homotopy sphere $A^4$ is homeomorphic and hence diffeomorphic to $S^4$ since $\Gamma_4 = 0$.

The remaining propositions combine to give a classification of manifolds of the form $E(A^n) + U^{n+k}$.

**Proposition 2.** $E(A^n)$ and $E(B^n)$ are diffeomorphic if and only if $A^n \equiv \pm B^n \mod \Phi_n^{k+1}$.

The proof of Proposition 2 is similar to the proofs of Lemmas 5 and 6 of [1]. R. Schultz informs me that he has also proved Proposition 1 and Proposition 2.
PROPOSITION 3. If \( E(A^n) + U^{n+k} \) is diffeomorphic to \( E(B^n) \), where \( A^n, B^n, U^{n+k} \) are homotopy spheres, then \( E(A^n) \) and \( E(B^n) \) are diffeomorphic.

The proof of Proposition 3 is similar to the proof of Lemma 3 of [1].

PROPOSITION 4. Let \( A^n, U^{n+k} \) be homotopy spheres such that \( 1 \leq k < n - 1 \). Then, \( E(A^n) + U^{n+k} \) is diffeomorphic to \( E(A^n) \) if and only if there exists an element \( \beta \in \pi_k(SO_{n-1}) \) such that

\[
U^{n+k} = \tau_{n,k}(A^n \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta).
\]

The proof of Proposition 4 is similar to the proof of Theorem 3.1 of [2] except that the proof here is a bit more complicated since there are two pairings involved rather than just the pairing \( \tau_{n,k} \).

Now we give the construction of the function \( \sigma'_{n,k} \) of (1) in §1. Let \( \gamma: S^{n-1} \to SO_k \) and \( \beta: S^k \to SO_n \) be differentiable maps that represent elements in \( \pi_{n-1}(SO_k) \) and \( \pi_k(SO_n) \), respectively. We can assume that \( \beta \) maps the southern hemisphere \( D_k^k \) of \( S^k \) into the identity of \( SO_n \). Define diffeomorphisms \( \lambda_\gamma \) and \( \mu_\beta \) of \( S^{n-1} \times S^k \) by writing, for each \( (u, v) \in S^{n-1} \times S^k \),

\[
\lambda_\gamma(u, v) = (u, s\gamma(u) \cdot v) \quad \text{and} \quad \mu_\beta(u, v) = (\beta(v) \cdot u, v);
\]

here the dot denotes the action of the rotation group on the sphere and \( s \) denotes the natural inclusion of \( SO_k \) in \( SO_{k+1} \). It is clear that \( \lambda_\gamma(S^{n-1} \times D_k^k) = S^{n-1} \times D_k^k \) and hence it follows that the diffeomorphism \( \lambda_\gamma^{-1}\mu_\beta\lambda_\gamma \) of \( S^{n-1} \times S^k \) is the identity on \( S^{n-1} \times D_k^k \). Now if \( B^{n+k} \) is an \((n+k)\)-disk in the interior of \( D^n \times S^k \), then it follows that the diffeomorphism \( \lambda_\gamma^{-1}\mu_\beta\lambda_\gamma \) can be extended to a diffeomorphism of \( D^n \times S^k - \text{Interior } B^{n+k} \). The diffeomorphism induced on the \((n+k-1)\)-sphere \( \partial B^{n+k} \) determines an element \( \sigma'_{n,k}(\gamma, \beta) \) of \( \theta_{n+k} \), and it is not hard to show that this element depends only on the homotopy classes of \( \gamma \) and \( \beta \). In fact, \( \sigma'_{n,k}(\gamma, \beta) \) is the obstruction to extending \( \lambda_\gamma^{-1}\mu_\beta\lambda_\gamma \) to a diffeomorphism of \( D^n \times S^k \). Since obstructions are additive with respect to compositions and

\[
\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma = (\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma)(\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma),
\]

the correspondence \( (\gamma, \beta) \mapsto \sigma'_{n,k}(\gamma, \beta) \) is linear in \( \beta \).

PROPOSITION 5. Let \( A^n \) and \( U^{n+k} \) be homotopy spheres such that \( 1 \leq n - 3 \leq k < 2n - 3 \). Then, \( E(A^n) + U^{n+k} \) is diffeomorphic to \( E(A^n) \) if and only if there exists an element \( \beta \in \pi_k(SO_n) \) such that \( U^{n+k} = \chi_{n}(\beta) \).

Now Theorem 2 follows by applying Propositions 1, 2, and 5, noting that \( \Phi^{k+1}_n = \theta_n \) for \( k \geq n - 3 \).

We conclude with some remarks on the case where \( k > n + 2 \). Proposition 1 is not true in this case. For example let \( \Sigma^k \) denote the non-
zero element of $\theta_16 \cong \mathbb{Z}_2$. It is known that $\Sigma^{16}$ does not embed in $R^{29}$ with a trivial normal bundle [3, Theorem 1.3]. Suppose that the conclusion of Proposition 1 is true for $S^{12} \times \Sigma^{16}$; that is, suppose that $S^{12} \times \Sigma^{16}$ is diffeomorphic to $(A^{12} \times S^{16}) + U^{28}$ for homotopy spheres $A^{12}$ and $U^{28}$. It is well known that $A^{12} \times S^{16}$ is diffeomorphic to $S^{12} \times S^{16}$ and hence it follows that $S^{12} \times \Sigma^{16}$ and $S^{12} \times S^{16}$ are diffeomorphic up to a point. This implies that $\Sigma^{16}$ embeds in $R^{29}$ with a trivial normal bundle, a contradiction. On the other hand if $k > n + 2$, then the characteristic class $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_{k+2})$.

Define homomorphisms $\eta_\alpha: \theta_k \to \theta_{n+k-1}$ and $\eta'_\alpha: \theta_{k+1} \to \theta_{n+k}$ by writing

$$\eta_\alpha(S^k) = c_{k,n-1}(S^k \otimes \alpha) \quad \text{and} \quad \eta'_\alpha(S^{k+1}) = c_{k+1,n-1}(S^{k+1} \otimes \alpha)$$

for $S^k \in \theta_k$ and $S^{k+1} \in \theta_{k+1}$, respectively. It follows from [2] that $\Phi'/\text{Kernel } \eta_\alpha$. Moreover, we can show that the number of distinct (nondiffeomorphic) differential structures on $E$ is not greater than the order of $\text{Kernel } \eta_\alpha/\Phi'$ times the order of $\theta_{n+k}/\text{Image } \eta'_\alpha$. We plan to give the explicit computation at a later date. Finally, it follows from Munkres [6] that the concordance classes of differential structures on $E$ are in a one-to-one correspondence with

$$\theta_n \oplus (\text{Kernel } \eta_\alpha) \oplus (\theta_{n+k}/\text{Image } \eta'_\alpha).$$

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