ASYMPTOTIC PROPERTIES OF ENTIRE FUNCTIONS
EXTREMAL FOR THE COUP THEOREM

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Let $f(z)$ be an entire function of order $p<1$. The classical “cos $\pi \rho$ theorem” of Valiron and Wiman [4, pp. 40, 51] asserts that if

$$\mu(r) = \min_{|z|=r} |f(z)|, \quad M(r) = \max_{|z|=r} |f(z)|,$$

then, given $\epsilon>0$, the inequality

$$(1) \quad \log \mu(r) > (\cos \pi \rho - \epsilon) \log M(r)$$

holds for a sequence $r=r_n \to +\infty$.

We consider those functions $f(z)$ for which (1) is the best possible inequality, and discuss the global asymptotic behavior of such functions.

**THEOREM 1.** Let $f(z)$ be an entire function of order $\rho$ ($0 \leq \rho < 1$), and suppose

$$(2) \quad \log \mu(r) \leq [\cos \pi \rho + \epsilon(r)] \log M(r)$$

where $\epsilon(r) \to 0$ as $r \to \infty$.

Then there exists a set $E$ of logarithmic density zero and a slowly varying function $\psi(r)$ such that

$$(3) \quad \log M(r) = r\psi(r) \quad (r \in E),$$

$$(4) \quad n(r, 0) = [\sin \pi \rho/\pi + o(1)] r\psi(r) \quad (r \to \infty, r \in E)$$

(where, as usual, $n(r, 0)$ denotes the number of zeros of $f(z)$ in $|z| \leq r$),

$$(5) \quad \log \mu(r) = [\cos \pi \rho + o(1)] r\psi(r) \quad (r \to \infty, r \in E \cup H),$$

where $H$ has (linear) density zero.

Further, there exists a real-valued function $\theta(r)$ such that if $k>1$ and $\delta>0$ are given and $v(r)$ denotes the number of zeros of $f(z)$ in the region

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2 A function $\psi(r)$ is said to vary slowly if it is defined and positive for all $r>r_0$ and satisfies $\lim_{r \to \infty} \psi(r)/\psi'(r) = 1$ ($0<\sigma<\infty$). For a useful discussion of the properties of such functions see, for example, [9, p. 419]. For a discussion of linear and logarithmic densities see [4, p. 5].
\{z: k^{-1}r \leq |z| \leq kr, \delta \leq \arg z - \theta(r) \leq \pi\},

then

(6) \quad \nu(r) = o(r^\psi(r)) \quad (r \to \infty, r \in E).

The function \theta(r) oscillates slowly outside of \( E \), in the sense that if \( k>1 \) and \( \epsilon>0 \) are given, then

(7) \quad |\theta(t) - \theta(r)| < \epsilon \quad (r > r_0(\epsilon, k), r \in E)

holds for all \( t \) in the interval \( k^{-1}r \leq t \leq kr \).

The content of the conclusions (3)-(7) can be expressed more intuitively if we say that on almost all long intervals, \( f(z) \) behaves like a Lindelöf function of order \( \rho \) [12, p. 18]. Indeed, it is not difficult to see that Theorem 1 implies that the asymptotic expansion

(8) \quad \log |f(te^{i(\theta+\phi)})| = [\cos \phi + o(1)]\psi(r)r^\rho

\quad (k^{-1}r \leq t \leq kr, t \in H, \theta \leq \pi),

is valid (uniformly in \( t \) and \( \phi \)) as \( r \to \infty \) outside \( E \), where \( \phi_0 = \theta(r) - \pi \), \( k>1 \) is a given constant, and \( H \) is the set of density zero given in Theorem 1.

Recent examples\(^4\) of W. K. Hayman [10] show that some exceptional set \( E \) must be present in Theorem 1; when coupled with Theorem 2 below, they also show that even in the important special case when all the zeros of \( f(z) \) are negative, \( E \) cannot be replaced by a set of linear density 0.

Theorem 1 may be compared to recent results of Kjellberg [11], Essén [7], Essén-Ganelius [8], and Anderson [1]. These authors consider (2) from another point of view; in particular, \( \rho \) can be any number, \( 0 < \rho < 1 \) (not necessarily the order of \( f(z) \)), but on the other hand, \( \epsilon(r) \) must satisfy some condition such as

\[
\limsup_{R_1, R_2 \to \infty} \int_{R_1}^{R_2} \frac{\epsilon(r) \log M(r)}{r^{1+\rho}} \, dr < M < \infty.
\]

Their conclusion, that \( \log M(r)/r^\rho \) tends to a limit \( \alpha \) (\( 0 \leq \alpha \leq \infty \)) as \( r \to \infty \) (with no need to avoid an exceptional set), is also of a different nature than that deduced here.

1. Outline of the proof. Let \( f(z) \) satisfy the hypotheses of the theorem. We can assume that \( f(0) = 1 \), and write

\(^4\) Hayman's examples are valid only if \( \rho = 1/2 \), but he comments that this case is probably typical.
(1.1) \[ f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\alpha_n} \right), \quad F(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{|\alpha_n|} \right). \]

Since

(1.2) \[ \log |F(-r)| + \log F(r) \leq \log \mu(r) + \log M(r) \]

[4, p. 40], it follows at once that \( F(z) \) also satisfies the hypotheses of Theorem 1, and hence a theorem of P. D. Barry [3] yields that

\[ \log |F(-r)| = [\cos \wp + o(1)] \log F(r) \quad (r \to \infty, r \in G^*), \]

where \( G^* \) has logarithmic density one. It is not hard to see, using (1.2) and hypothesis (2) again, that

(1.3) \[ \log M(r) \sim \log F(r) \quad (r \to \infty, r \in G^*). \]

An easy extension of Theorem 2 of [2] now shows that from (1.3) follows

(1.4) \[ \nu(r) = o(\log M(r)) \quad (r \to \infty, r \in G^*). \]

We next establish that \( G^* \) can be replaced by a subset \( G_* \) having the following crucial properties: there are sequences \( \{\alpha_n\}, \{\beta_n\} \) and a set \( H \) of (linear) density zero such that

\[ G_* = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] - H \quad (\alpha_n \to \infty, \beta_n/\alpha_n \to \infty) \]

has logarithmic density one, and, if \( k > 1 \) is given, then

\[ [k^{-1}\alpha_n, k\beta_n] \subseteq G^* \cup H \quad (n > n_0(k)). \]

The exceptional set \( E \) which appears in the statement of Theorem 1 is the complement of \( G = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] \).

In view of elementary properties of sets of linear density zero, it is easy to see that (1.3) holds with \( G^* \) replaced by \( G \), and so it suffices to prove (3)--(5) for \( F(z) \). The argument hinges now on a suitable generalization (to allow exceptional sets of logarithmic density zero) of the following theorem [6], which is one form of a complement to some classical results of Titchmarsh [13] and Bowen and Macintyre [5].

**Theorem 2.** Let \( F(z) \) be an entire function of the form (1.1), and suppose

\[ \frac{\log |F(-r)|}{\log F(r)} \to \alpha \quad (r \to \infty, r \in H), \]

where \( H \) is of (linear) density zero.
Then $-1 \leq \alpha \leq 1$, and
\[
\log F(r) = r^\psi(r),
\]
\[
n(r,0) = \left[ \frac{\sin \pi \rho}{\pi} + o(1) \right] r^\psi(r) \quad (r \to \infty),
\]
\[
\log |F(-r)| = [\cos \pi \rho + o(1)] r^\psi(r) \quad (r \to \infty, r \in H),
\]
where $\rho$ is determined by
\[
\cos \pi \rho = \alpha \quad (0 \leq \rho \leq 1),
\]
and $\psi$ is a slowly varying function.

Finally, (6) follows from (1.4) and (3), and (7) is an easy consequence of (6) (cf. [2, Corollary 1]).

Conclusion (3) of Theorem 1 implies that $f(z)$ has regular growth in the sense of Valiron. Analogues of Theorem 1, valid for functions of irregular growth, can also be derived from these methods.

References

3. P. D. Barry, Some theorems related to the $\cos \pi \rho$ theorem, A. J. Macintyre memorial volume, Ohio Univ. Press (to appear).
10. W. K. Hayman, Some examples related to the $\cos \pi \rho$-theorem (to appear).

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