W-SYSTEMS ARE THE WALSH FUNCTIONS

BY DANIEL WATERMAN

Communicated by Henry Helson, July 15, 1968

Let \( \{\phi_n\} \), \( n = 0, 1, 2, \ldots \), be a system of functions on \((0, 1)\) with \( \phi_0 = 1 \). For \( n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p} \), with \( 0 \leq k_1 < k_2 < \cdots < k_p \), we set

\[
\psi_n = \phi_{k_1} \cdot \phi_{k_2} \cdot \ldots \cdot \phi_{k_p}.
\]

If \( \{\psi_n\} \) is an orthogonal system on \((0, 1)\), it is called a \( W \)-system \([1, \text{pp. 185-196}] \) after the Walsh system, \( \{w_n\} \), which is formed from the Rademacher functions, \( \{r_n\} \), in this manner. It is generally assumed that the system \( \{\psi_n\} \) is equinormed, i.e., there is a constant \( K \) such that

\[
(*) \quad \int_0^1 |\psi_n|^2 = K \quad \text{for} \quad n \geq n_0.
\]

The study of these systems has shown that they are essentially of two types depending on whether or not we assume

\[
(**) \quad |\phi_n(x)| \leq 1 \text{ a.e.} \quad \text{for every} \quad n.
\]

When this assumption is made, the results obtained on \( W \)-systems parallel those for the Walsh system. In the other case, the behavior may resemble that of systems generated by the strongly lacunary trigonometric sequences \( \{2^{1/2} \cos m_n x\} \) and \( \{2^{1/2} \sin m_n x\} \) with \( m_{n+1}/m_n \geq 3 \) \([1, \text{pp. 190-191}] \) and \([2, \text{pp. 208-209}] \).

We will restrict our consideration to the systems satisfying (*) and (**), and we will refer to them simply as \( W \)-systems.

In very general terms our results may be stated as follows.

A result concerning the a.e. convergence or summability of a Walsh series, \( \sum c_n w_n \), implies the corresponding result for the \( W \)-series, \( \sum c_n \psi_n \).

We will state more precise results shortly.

From our first lemma we conclude that we may assume \( |\phi_n(x)| = 1 \) a.e. without loss of generality.

**Lemma 1.** To any system \( \{\phi_n\} \) on \((0, 1)\) with

\[
\int |\phi_i|^2 = \int |\phi_i \phi_j|^2 = K \quad \text{for} \quad i \neq j
\]

---

1 Supported by National Science Foundation Grant GP7358.
and

\[ |\phi_n(x)| \leq 1 \text{ a.e. for all } n, \]

there corresponds a measurable set \( E \subset (0, 1) \), \( m(E) = K \), such that, for every \( n \), \( |\phi_n| = 1 \) on \( E \) and \( |\phi_n| = 0 \) a.e. on \( E^c \).

It is not difficult to see that this implies that the system \( \{\psi_n\} \) lives on a set \( E \) of measure \( K \) in the sense that

(i) \( |\psi_n| = 1 \) on \( E \) for every \( n \),
(ii) \( m(\{|\psi_n| \neq 0\} \cap E^c) > 0 \) for only finitely many \( n \),
(iii) \( \{\psi_n\} \) is orthogonal relative to \( E \).

Alexits has called a system \( \{\phi_n\} \) multiplicatively orthogonal if \( \phi_n = 0 \) for \( n > 0 \) and strongly multiplicatively orthogonal if \( \{\psi_n\} \) is orthogonal \([1, \text{ pp. 186-187}]\). We note that under the hypotheses of Lemma 1, these notions coincide relative to \( E \).

If \( A \) is a 1-1 measure preserving map of \((0, 1)\) onto itself taking \((0, K)\) into \( E \), we see that \( \{\psi_n \circ A(Kx)\} \) is a \( W \)-system on \((0, 1)\) and \( |\psi_n \circ A(Kx)| = 1 \) a.e. for all \( n \). Thus we can reduce the study of the original system on \( E \) to the study of a \( W \)-system living on \((0, 1)\).

We will assume, henceforth, that \( |\psi_n(x)| = 1 \) a.e. for all \( n \).

Let us consider the sets on which \( \phi_1, \ldots, \phi_k \) are of constant sign. For any integer \( t \in [0, 2^k - 1] \), we have a unique representation

\[ t = a_k2^0 + \cdots + a_12^{k-1} \]

with \( a_v = 0 \) or 1. Set

\[ E_t^k = \{x: \phi_v(x) = e^{i\pi a_v}, \, v = 1, \ldots, k\}. \]

Then \((0, 1) = \bigcup_{t=0}^{2^k-1} E_t^k\), modulo a null set, and the sets \( E_t^k \) are pairwise disjoint and measurable. We have, indeed,

**Lemma 2.** \( m(E_t^k) = 1/2^k \) for all \( k \) and \( t = 0, 1, \ldots, 2^k - 1 \).

We now define a function on \((0, 1)\) by means of the dyadic representation of its values, \( y(x) = \alpha_1\alpha_2\cdots \), with

\[ \alpha_v = (1/i\pi) \log \phi_v(x), \, v = 1, 2, 3, \ldots. \]

We see at once that, except for those points for which \( \alpha_v = 1 \) or \( \alpha_v = 0 \) from some index onward, \( y(x) = a_1a_2a_3\cdots, \, a_v = 0 \) or 1, if and only if \( x \in E_t^k, \, t = 2^k(a_1\cdots a_k) \) for every \( k \). This exceptional set, as well as the set for which \( \alpha_v \neq 0 \) or 1 for some \( v \), is easily seen to be a set of measure zero.

We have the following result.
LEMMA 3. For every measurable $E \subseteq (0, 1)$, $y^{-1}(E)$ is measurable and $m(y^{-1}(E)) = m(E)$.

A sequence $\{f_n\}$ of bounded measurable functions is said to be maximal if there is a set $Z$ of measure zero such that if $f_n(x_1) = f_n(x_2)$ for every $n$ and $x_1, x_2 \in Z$, then $x_1 = x_2$. Rényi [3] showed that maximality is sufficient to imply that the system

$$\{f_1^{m_1}f_2^{m_2} \cdots f_n^{m_n}\},$$

$m_1 = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots$, is closed in $L^2$. We [4] have shown that maximality is also necessary.

Clearly the sequence $\{\psi_n\}$ is maximal if and only if $y$ is almost everywhere 1-1. We have further

LEMMA 4. If $\{\phi_n\}$ is maximal, there is a metric automorphism $\eta$ on $(0, 1)$ such that $\eta(x) = y(x)$ a.e.

By a metric automorphism of a set we mean a 1-1 measure preserving mapping of the set onto itself.

Applying these results to $W$-systems we have

THEOREM 1. If $\{\psi_n\}$ is a $W$-system, then the following conditions are equivalent:

(i) $\{\psi_n\}$ is complete.
(ii) There is a metric automorphism $\eta$ of $(0, 1)$ such that $\psi_n(x) = w_n \circ \eta(x)$ a.e. for every $n$.

We see then that for complete $W$-systems the study of the series $\sum c_n\psi_n$ can be replaced by the study of the Walsh series $\sum c_nw_n$. We note further that for every $f \in L^p$, $p \geq 1$, $f \sim \sum c_n\psi_n$, $g = f \circ \eta^{-1} \in L^p$ and $g \sim \sum c_nw_n$. Thus most results of an almost everywhere nature concerning the Walsh-Fourier series of functions in $L^p$ are extendable to $W$-Fourier series.

This is still the case for some types of results even if $\{\psi_n\}$ is not complete, for in this case, if $g \in L^p$, $p \geq 1$, $g \sim \sum c_nw_n$, then $h = g \circ y \in L^p$ and $h \sim \sum c_n\psi_n$.

Since Stein [5] has shown that there is an $L^1$ function whose Walsh-Fourier series diverges a.e., we have

THEOREM 2. For any $W$-system there is an integrable function whose $W$-Fourier series diverges a.e.

If $f \in L^2$, $f \sim \sum c_n\psi_n$, then there is an $f^* \in L^2$, $f^* \sim \sum c_nw_n$. Since Billard [6] has solved Lusin's problem for Walsh-Fourier series we have...
Theorem 3. The W-Fourier series of an $L^2$ function converges a.e.

Our final result concerns a peculiar type of invariance under rearrangement. We state the results for Walsh series. We do not believe that this has been noted previously.

Suppose $\{r_{ni}\}, i = 1, 2, \ldots$, is a rearrangement of the Rademacher system. This induces a rearrangement of the Walsh system by means of the relation

$$w_{m_i} = r_{n_{i_1}} \cdot r_{n_{i_2}} \cdot \ldots \cdot r_{n_{i_k}}$$

where

$$m_i = 2^{n_{i_1}} + \ldots + 2^{n_{i_k}}.$$ 

Such rearrangements we call coherent. We have the following result.

Theorem 4. If $\{w_{m_n}\}$ is a coherent rearrangement of the Walsh system, then the almost everywhere convergence and summability behaviors of the series $\sum c_n w_n$ and $\sum c_n w_{m_n}$ are the same.

References


Wayne State University, Detroit, Michigan 48202