CROSS SECTIONALLY SIMPLE SPHERES

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J. W. Alexander [1] suggested that a 2-sphere \( S \) in \( E^3 \) is tame if each horizontal cross section is either a point or a simple closed curve. It is not clear whether he presumed that his proof was valid for non-polyhedral spheres, but his proof implies that there is a homeomorphism \( h \) of \( E^3 \) onto itself which is invariant on horizontal planes and which takes \( S \) onto a round 2-sphere. Bing [6] has described a non-polyhedral 2-sphere \( S \) for which there is no such homeomorphism \( h \).

In this paper we give a proof of Alexander's conjecture. The proof, however, is not elementary as it depends indirectly on Dehn's Lemma [8], Bing's Side Approximation Theorem [2], and Bing's Characterization of tame spheres with homeomorphic approximations in their complementary domains [4].

We assume that \( S \) lies exactly between the planes \( z = 1 \) and \( z = -1 \) and we let \( J_t = S \cap \{(x, y, z)|z = t\} \) be the horizontal cross section of \( S \) at the \( z = t \) plane. Note that \( J_t \) is a simple closed curve for \( -1 < t < 1 \) and \( J_{-1} \), \( J_1 \) are points. We let \( D_t \) be the disk \( J_t \) bounds in the \( z = t \) plane. The \( \epsilon \)-neighborhood of a set \( X \) is denoted by \( N(X, \epsilon) \), \( \text{Diam} A \) is the diameter of \( A \), and \( S^1 \) stands for the standard 1-sphere. If \( -1 < \alpha < \beta < 1 \) and \( h \) is a homeomorphism of \( S^1 \times [\alpha, \beta] \) into \( \text{Int} S \) such that

1. \( h(y \times [\alpha, \beta]) \) is a vertical line segment for \( y \in S^1 \), and
2. \( h(S^1 \times t) \) lies in the plane \( z = t \) for \( t \in [\alpha, \beta] \),

then \( A(h, t) \) denotes the annulus in the \( z = t \) plane bounded by \( h(S^1 \times t) \) and \( J_t \), \( S(\alpha, \beta) \) denotes the annulus \( S \cap \{(x, y, z)|\alpha \leq z \leq \beta\} \) and \( T(h) \) denotes the torus \( h(S^1 \times [\alpha, \beta]) \cup A(h, \alpha) \cup A(h, \beta) \cup S(\alpha, \beta) \).

**Lemma 1.** If \( t \in (-1, 1) \) and \( \epsilon > 0 \) then there are rational numbers \( \alpha \) and \( \beta \) and a homeomorphism \( h: S^1 \times [\alpha, \beta] \rightarrow \text{Int} S \) such that

1. \( -1 < \alpha < t < \beta < 1 \),
2. \( h(y \times [\alpha, \beta]) \) is a vertical line segment for each \( y \in S^1 \),
3. \( h(S^1 \times r) \) lies in the horizontal plane \( z = r \) for \( r \in [\alpha, \beta] \),
4. \( T(h) \) lies in an \( \epsilon \)-neighborhood of \( J_t \), and
5. \( h(S^1 \times t) \) is homeomorphically within \( \epsilon \) of \( J_t \).

**Proof.** There is a simple closed curve \( J \) in the \( z = t \) plane such that \( J \subset \text{Int} S \), \( J \) is homeomorphically within \( \epsilon \) of \( J_t \), and the annulus \( A \) bounded by \( J \) and \( J_t \) in the \( z = t \) plane lies in \( N(J_t, \epsilon) \). \( J \) may be moved
slightly in the vertical direction so there is a $\delta > 0$ and a homeomorphism $g: S^1 \times [t-\delta, t+\delta] \to \text{Int } S$ such that $g(y \times [t-\delta, t+\delta])$ is a vertical line segment for $y \in S^1$, $g(S^1 \times r)$ lies in the $z=r$ plane for $r \in [t-\delta, t+\delta]$ and $g(S^1 \times I) = J$. Since the annulus $A(g, t)$ lies in $N(J, \epsilon)$ and $\lim_{r \to t} A(g, r) = A(g, t)$, there are rational numbers $\alpha$ and $\beta$ such that $t-\delta < \alpha < t < \beta < t+\delta$ and $A(g, r) \subseteq N(J, \epsilon)$ if $r \in [\alpha, \beta]$. Take $h = g|S^1 \times [\alpha, \beta]$.

The following lemma is an easy consequence of the Tietze Extension Theorem and the fact that small subsets of $S$ lie in small subdisks on $S$.

**Lemma 2.** If $D$ is a 2-cell and $\epsilon > 0$ then there is a $\delta > 0$ such that if $f$ is a map of $\text{Bd } D$ into a $\delta$-subset of $\text{Int } S$ then $f$ may be extended to $D$ so that $f(D)$ lies in an $\epsilon$-subset of $\text{Cl}(\text{Int } S)$.

To establish that $S$ is tame from $\text{Int } S$ we show that $\text{Int } S$ is 1-ULC and use Bing's characterization of tame 2-spheres in $E^3$ [3]. That $\text{Int } S$ is 1-ULC is an easy consequence of Lemmas 2 and 3.

**Lemma 3.** If $f$ is a map of 2-cell $D$ into $\text{Cl}(\text{Int } S)$ such that $f(\text{Bd } D) \subseteq \text{Int } S$ and $\epsilon > 0$ then there is a map $g: D \to \text{Int } S$ such that $f|\text{Bd } D = g|\text{Bd } D$ and $g(D) \cap (f(D), \epsilon)$.

**Proof.** The map $f$ is adjusted in three steps to obtain $g$. In Step I, $f$ is adjusted so that $f(D)$ misses the points $J_1$ and $J_{-1}$. In Step II, $f$ is further adjusted so that $f(D) \cap S$ is 0-dimensional and $f(D) \cap J_r = \emptyset$ if $r$ is a rational number. The map $f$ is altered in Step III so that $f(D) \subseteq \text{Int } S$.

**Step I.** There is a $\delta > 0$ such that $\text{Diam } D_{1-\delta} < \epsilon/3$, $\text{Diam } D_{-1+\delta} < \epsilon/3$, $D_{1-\delta}$ separates $f(\text{Bd } D)$ from $J_1$ on $f(D)$, and $D_{-1+\delta}$ separates $f(\text{Bd } D)$ from $J_{-1}$ on $f(D)$. In two applications of the Tietze Extension Theorem, as indicated in [7, Lemma 1], we may adjust $f$ to obtain a map $f_1: D \to \text{Cl}(\text{Int } S)$ such that $f|\text{Bd } D = f_1|\text{Bd } D$, $f_1(D) \subseteq (f(D), \epsilon/3)$, $J_1 \subseteq f_1(D)$, and $J_{-1} \subseteq f_1(D)$.

**Step II.** Since for $-1 < r < 1$, $J_r$ is a tame simple closed curve, it follows from the techniques of [5] that there is a map $f_2: D \to \text{Cl}(\text{Int } S)$ such that $f_2|\text{Bd } D = f_1|\text{Bd } D$ this $D$, $f_2(D) \subseteq (f_1(D), \epsilon/3)$, $f_2(D) \cap S$ is 0-dimensional, and $J_r \cap f_2(D) = \emptyset$ if $r$ is rational.

**Step III.** For $-1 < t < 1$ there is a $\delta_t > 0$ such that if $J$ is homeomorphically within $\delta_t$ of $J_t$ then each $\delta_t$ subset of $J$ lies in an $\epsilon/18$ arc in $J$. Since $f_2(D) \cap S$ is compact and 0-dimensional, for each $-1 < t < 1$ there is a collection $E_t$ of disjoint open sets in $E^3$ covering $f_2(D) \cap S$ such that if $A \in E_t$ then $\text{Diam } A < \delta_t$ and $A \cap f(\text{Bd } D) = \emptyset$. Let $\lambda_1$ be a positive number which is less than the distance between $f_2(D) - \bigcup E_t$.
and $J_t$ and which is less than $\varepsilon/36$. By Lemma 1 there are rational numbers $\alpha_t$ and $\beta_t$ and a homeomorphism $h_t : S^1 \times [\alpha_t, \beta_t] \to \text{Int } S$ such that

1. $-1 < \alpha_t < t < \beta_t < 1$,
2. $h_t(y \times [\alpha_t, \beta_t])$ is a vertical line segment for each $y \in S^1$,
3. $h_t(S^1 \times r)$ lies in the horizontal plane $z = r$ for $r \in [\alpha_t, \beta_t]$,
4. $T(h_t)$ lies in an $\lambda$-neighborhood of $J_t$, and
5. $h_t(S^1 \times t)$ is homeomorphically within $\varepsilon/18$ of $J_t$.

A finite number of the tori $\{T(h_t)\} \ (-1 < t < 1)$ suffice to cover $f_2(D) \cap S$. This finite collection of tori may be cut apart using horizontal planes $z = r$ with $r$ rational to obtain a new finite collection $E$ of disjoint tori which also cover $f_2(D) \cap S$. For each $T \in E$ there exist a number $t$ and rational numbers $u$ and $v$ such that $\alpha_t < u < v < \beta_t$ and $T = h_t(S^1 \times [u, v]) \cup A(h_t, u) \cup A(h_t, v) \cup S(u, v)$.

We next show that each component $K$ of $f_2(D) \cap (T - S(u, v))$ is contained in the interior of an $\varepsilon/3$-disk in $T - S(u, v)$. By (4) there is an open set in $E_t$ which contains $K$; consequently, $\text{Diam } K < \delta_t$. Let $K' = \{x \in h_t(S^1 \times t) | \text{for some } y \in S^1, x = h_t(y \times t) \}$ and $K \cap h_t(y \times [u, v]) \neq \emptyset$. It follows that $\text{Diam } K' < \delta_t$ so there is an $\varepsilon/18$-arc $B$ in $h_t(S^1 \times t)$ that contains $K'$. If $M = \{x \in S^1 | h_t(x \times t) \in B\}$ then the disk $E = \bigcup_{x \in M} h_t(x \times [u, v])$ is of diameter less than $\varepsilon/9$ and contains $K \cap h_t(S^1 \times [u, v])$. Since $\text{Diam } K \cap \text{Int } A(h_t, u) < \delta_t < \varepsilon/9$, $\delta_t < \text{Diam } h_t(S^1 \times u)$, and $A(h_t, u)$ lies in a horizontal plane, it follows that $K \cap \text{Int } A(h_t, u)$ does not separate $J_u$ from $h_t(S^1 \times u)$ on $A(h_t, u)$ and consequently lies in an $\varepsilon/9$-subdisk $E_u$ of $A(h_t, u) - J_u$. $E_u$ may be chosen so that $E_u \cap E = \text{Bd } E_u \cap \text{Bd } E$ is an arc. Similarly, there is an $\varepsilon/9$-subdisk $E_v$ of $A(h_t, v) - J_v$ that contains $K \cap A(h_t, v)$ so that $E_v \cap E = \text{Bd } E_v \cap \text{Bd } E$ is an arc. It follows that $E_u \cup E \cup E_v$ is an $\varepsilon/3$-subdisk of $T - S(u, v)$ that contains $K$.

Using techniques of the topology of $E^2$ it now follows that $(T - S(u, v)) \cap f_2(D)$ is covered with a finite collection $\mathcal{D}_T$ of disjoint $\varepsilon/3$-disks in $T - S(u, v)$. The collection $\mathcal{D} = \bigcup_{T \in \mathcal{D}_T} T$ is finite, disjoint, and such that $U_{F \in \mathcal{D}} F$ separates $f_2(\text{Bd } D)$ from $f_2(D) \cap S$ on $f_2(D)$. It follows from the Tietze Extension Theorem, as indicated in [7, Lemma 1], that there is a map $g : D \to \text{Int } S$ such that $g|\text{Bd } D = f_2|\text{Bd } D$ and $g(D) \subset N(f_2(D), \varepsilon/3)$.

From Steps I, II, and III we have $f|\text{Bd } D = f_1|\text{Bd } D = f_2|\text{Bd } D = g|\text{Bd } D$ and $g(D) \subset N(f(D), \varepsilon)$.

That $S$ is tame from $\text{Int } S$ now follows from Lemma 2, Lemma 3 and Bing's 1-ULC characterization of tame surfaces. Similar techniques are employed to show that $S$ is tame from $\text{Ext } S$. Thus we have proved the following theorem suggested by Alexander [1].
Theorem. A 2-sphere $S$ in $E^3$ is tame if each horizontal cross section of $S$ is either a simple closed curve or a point.

The author has recently learned that Norman Hosay has also given a proof of this theorem.

References