CROSS SECTIONALLY CONTINUOUS SPHERES IN $E^3$

BY L. D. LOVELAND

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W. T. Eaton [5] and Norman Hosay [6] independently solved a problem raised by Alexander [1] when they proved that cross sectionally simple spheres in $E^3$ are tame. A 2-sphere $S$ in $E^3$ is cross sectionally simple if the intersection of $S$ with each horizontal plane $P_t = \{(x, y, z)|z=t\}$ is either empty, a point, or a simple closed curve. The purpose of this note is to show that Eaton's proof, adjusted slightly to incorporate recent results by J. W. Cannon [4], actually shows that cross sectionally continuous spheres are tame. A 2-sphere $S$ in $E^3$ is cross sectionally continuous if each $S \cap P_t$ is a locally connected continuum. The question about the tameness of cross sectionally connected spheres remains open [2].

As in [5] we assume that the cross sectionally continuous sphere $S$ intersects $P_t$ if and only if $-1 \leq t \leq 1$, and we denote the continuum $P_t \cap S$ by $J_t$.

There are two observations to make before we proceed to details of the adjustment of Eaton's proof. First we observe that at most countably many $J_t$ fail to be simple closed curves. This is because a locally connected continuum that is not a simple closed curve must contain a simple triod, and $S$ cannot contain uncountably many disjoint triods [7]. The second observation is that a tame nondegenerate continuum $J_t$ on $S$ is a taming set; that is, each 2-sphere containing $J_t$ and locally tame modulo $J_t$ is tame [4]. In the following paragraphs, we indicate briefly how to incorporate these two observations into Eaton's proof to establish

**Theorem 1.** Cross sectionally continuous 2-spheres in $E^3$ are tame.

Let $R$ be a countable subset of $[-1, 1]$ such that

1. if $J_t$ is not a simple closed curve, then $t \in R$ and
2. $R$ contains a subset $X$ dense in $[-1, 1]$ such that for $t \in X$, $J_t$ is a simple closed curve, and let $Q = [-1, 1] - R$.

One may think of $R$ as the set of rational numbers in $[-1, 1]$. Lemma 1 of [5] remains valid for cross sectionally continuous spheres as long as $t$ is restricted to $Q$, and Lemma 2 of [5] is retained as it stands.

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The three steps given in the proof of Lemma 3 of [5] require little change here. Eaton's Step I is used here for each degenerate $J_i$ ($i = 1, -1$). Otherwise we skip Step I. To accomplish Step II we use Cannon's result that each $J_r$ ($r \in R$) is a taming set, together with the techniques of [3]. In Step III we again restrict $t$ to $Q$. No other changes are required.

Actually we have outlined a proof for the following more general theorem.

**Theorem 2.** If each horizontal cross section of a 2-sphere $S$ in $E^3$ is connected and at most countably many of these cross sections fail to be locally connected, then $S$ is tame.

**References**


Utah State University, Logan, Utah 84321 and
University of Utah, Salt Lake City, Utah 84112