


14. ———, *Singular integrals are Perron integrals of a certain type* (to appear).


UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616

ON GENERALIZED COMPLETE METRIC SPACES

BY JAMES D. STEIN, JR.

Communicated by J. B. Diaz, October 3, 1968

The following remarks are of interest in connection with the research announcement [1]:

**Lemma.** A generalized metric space is the disjoint union of metric spaces such that each metric space is infinitely distant from every other metric space.

**Proof.** Note that $d(x, y) < \infty$ is an equivalence relation, and the equivalence classes obtained are metric spaces. Also, if the generalized space is complete, so is each metric space. Q.E.D.

Let $M = \bigvee_{\alpha \in A} M_\alpha$ denote the above partitioning. The Banach contraction principle becomes

**Proposition 1.** Let $T$ be a strict contraction of a generalized complete metric space $M = \bigvee_{\alpha \in A} M_\alpha$, $0 \leq q < 1$, $d(x, y) < \infty \Rightarrow d(Tx, Ty) \leq qd(x, y)$. For each $\alpha \in A$, $\exists \beta \in A$ such that $T(M_\alpha) \subseteq M_\beta$. There is a unique periodic point of order $n$ in each $M_\alpha$ such that $T^n(M_\alpha) \subseteq M_\alpha$.

**Proof.** Let $x, y \in M_\alpha$, $Tx \in M_\beta$. Then $d(x, y) < \infty \Rightarrow d(Tx, Ty) < \infty \Rightarrow Ty \in M_\beta$. Since $T^n$ is a strict contraction of the complete metric space $M_\alpha$, it has a unique fixed point, which is a periodic point of order $n$ for $T$. Q.E.D.

The local contraction principle becomes
PROPOSITION 2. Let $T$ be a local contraction $(d(x, y) \leq C \Rightarrow d(Tx, Ty) \leq qd(x, y))$ of a complete generalized metric space. For each $\alpha \in A$, $x, y \in M_\alpha$, define $x \sim y$ iff $\exists x_0, \ldots, x_n \in M_0$ such that $x = x_0$, $y = x_n$, $d(x_i, x_{i+1}) \leq C$ for $0 \leq i \leq n-1$. Then $\sim$ is an equivalence relation on each $M_\alpha$; call the equivalence classes thus obtained $C$-components. $T$ maps each $C$-component of $M_\alpha$ into a $C$-component of some $M_\beta$. There is a unique periodic point of order $n$ in each $C$-component $N$ of $M_\alpha$ such that $T^n(N) \subseteq N$.

Proof. Clearly $x \sim y$ is an equivalence relation; and if $x, y \in M_\alpha$ and $x \sim y$, let $x = x_0, \ldots, x_n = y$ be the chain. Then $d(Tx_i, Tx_{i+1}) \leq qd(x_i, x_{i+1}) \leq C$, and so $Tx \sim Ty$ in some $M_\beta$. The remainder is Theorem 1.4 of Bonsall (or Edelstein) of On some fixed point theorems of functional analysis. Q.E.D.

Several of Edelstein’s and Rakotch’s results go over analogously.

Reference

1. J. B. Diaz and Beatriz Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305–309.

University of California, Los Angeles, California 90024