Let \( E^n \) denote Euclidean \( n \) space with norm \( \| \cdot \| \) and let \( C_h \) denote the space of continuous \( E^n \) valued functions on \([a - h, a]\), \( h > 0 \), with uniform norm \( \| \cdot \| \). For a function \( x(t) \) on \([a - h, b]\) and \( t \in [a, b] \) let \( x_t \) denote the function on \([a - h, a]\) whose value at \( \theta \) is \( x(t + \theta - a) \). Let \( f(t, \Psi) \) be a mapping from \([a, b] \times C_h \) into \( E^n \) and let \( M \) and \( N \) be linear operators from \( C_h \) to \( C_h \). In this paper we consider special cases of the boundary value problem

\[
\begin{align*}
(1) & \quad y' = f(t, y_t), \\
(2) & \quad M y_a + N y_b = 0, \quad b > a + h.
\end{align*}
\]

This is a nonlinear version of a problem posed by Cooke [1]. Other boundary value problems for functional differential equations have been studied recently by Grimm and Schmitt [3], Halanay [4], and Kato [6].

We treat the problem for bounded \( f \) and a restricted class of operators by initial value methods, that is, we seek to find an initial function \( q \in C_h \) such that a solution of the initial value problem (1) and

\[
(3) \quad y(t) = q(t), \quad a - h \leq t \leq a
\]
satisfies the boundary condition (2). Some functions \( f(t, y_t) \), not bounded, can be treated by approximation techniques and some nonhomogeneous boundary conditions can also be considered.

**Theorem 1.** Let \( f(t, \Psi) \) be a continuous bounded function from \([a, b] \times C_h \) into \( E^n \) and let \( M \) and \( N \) be \( n \times n \) matrices such that \( M + N \) is nonsingular. If \( \| (M + N)^{-1} N \| < 1 \), then there exists a solution of (1) and (2).

**Method of Proof.** We give a brief sketch of the method of proof of Theorem 1. The proofs of the other theorems are similar. Assume first that solutions of the initial value problem are unique. Let \( T: C_h \rightarrow C_h \) be defined as follows: for \( q \in C_h \), let \( Tq = x_b(q) \), that is, \( Tq \) is the segment at \( b \) of the solution of the initial value problem with
initial data \( q \). Let

\[ S(\alpha, L) = \{ q \mid q \in C_h, \|q\| \leq \alpha, \| q(\theta_1) - q(\theta_2) \| \leq L \| \theta_1 - \theta_2 \|, \theta_1, \theta_2 \in [a - h, a] \} \]

The operator

\[
F(q) = [I - (M + N)^{-1}(M + NT)]q
= -(M + N)^{-1}N(T - I)q
\]

is shown to have a fixed point \( q^* \in S(\alpha, L) \) for an appropriate choice of \( \alpha \) and \( L \). The solution of the initial value problem with initial data \( q^* \), satisfies the boundary condition (2). An approximation argument, essentially that of Kato [6], is then used to remove the uniqueness assumption.

In case the matrix \( M \) is invertible the boundary condition (2) can be written

\[ y(a) + Py(b) = 0. \]

If \( \|P\| < 1 \), then \( I + P \) is nonsingular [2, p. 62] but it is not necessarily the case that \( \|(I + P)^{-1}P\| < 1 \). The following theorem covers this case.

**Theorem 2.** Let \( f(t, \Psi) \) be a continuous bounded function from \( [a, b] \times C_h \) into \( \mathbb{E}^n \) and let \( P \) be an \( n \times n \) matrix. If \( \|P\| < 1 \), then the boundary value problem (1) (4) has a solution.

In the preceding theorems, the desired initial function was selected from a compact set \( S(\alpha, L) \). We can replace the matrix \( P \) in Theorem 2 by a linear operator but we no longer can select the initial condition from a predetermined compact set and thereby lose the approximation argument which allowed the assumption of unique solutions of the initial value problem (1) (3) to be avoided.

**Theorem 3.** Let \( f(t, \Psi) \) be a continuous bounded function from \( [a, b] \times C_h \) into \( \mathbb{E}^n \) and let \( P \) be a continuous linear operator from \( C_h \) into \( C_h \) with \( \|P\| < 1 \). Suppose solutions of the initial value problem are unique. Then there exists a solution of (1) (4).

**Theorem 4.** Let the hypotheses of Theorem 3 hold with the boundedness assumption on \( f(t, \Psi) \) replaced by a Lipschitz condition. If

\[ e^{L(b-a)}(b-a) < (1 - \|P\|)/L\|P\|, \]

there exists a unique solution of (1) (4).

Details will appear elsewhere.
REFERENCES


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