THE FIXED POINT INDEX AND ASYMPTOTIC FIXED
POINT THEOREMS FOR k-SET-CONTRACTIONS

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1. Introduction. In 1955 G. Darbo [6] defined the measure of
noncompactness, \( \gamma(A) \), of a bounded subset \( A \) of a metric space
\( (X, d) \): \( \gamma(A) = \inf \{ d > 0 \mid A \text{ can be covered by a finite number of sets}
\text{of diameter less than or equal to } d \} \). If \( (X, d) \) is a complete metric
space, Darbo shows that for any decreasing sequence of closed, non­
empty sets \( A_n \) with \( \gamma(A_n) \) approaching 0, \( \cap_{n \geq 1} A_n \) is compact and
nonempty. If \( X \) is a Banach space, Darbo also demonstrates the
crucial properties \( \gamma(A+B) \leq \gamma(A) + \gamma(B) \) and \( \gamma(\text{convex closure } A) = \gamma(A) \).

If \( G \) is a subset of the metric space \( X_1 \) and \( f \) is a continuous map
from \( G \) to a metric space \( X_2 \), Darbo calls \( f \) a \( k \)-set-contraction if
\( \gamma(f(A)) \leq k \gamma(A) \) for \( A \) bounded and \( A \subseteq G \). It is easy to show that
\( k \)-set-contractions with \( k < 1 \) are closed under composition and convex
sums. Darbo proves that if \( G \) is a closed, bounded convex subset of a
Banach space \( X \) and \( f: G \rightarrow G \) is a \( k \)-set-contraction, \( k < 1 \), then \( f \)
has a fixed point.

An important example of a \( k \)-set-contraction, \( k < 1 \), is a map of the
form \( U + C \), \( U \) a strict contraction (i.e. \( \| Ux - Uy \| \leq k \| x - y \|, \ k < 1 \) and
\( C \) a compact map, both defined on a subset \( G \) of a Banach space
\( X \). F. E. Browder and the author [5] have recently defined (as a
special case) a degree theory for mappings of the form \( I - U - C \), so
it is natural to ask if one can obtain a degree theory for mappings of
the form \( I - f \), \( f \) a \( k \)-set-contraction, \( k < 1 \). In fact we will define a
fixed point index for \( k \)-set-contractions on certain nice ANR’s, and
we will give direct generalizations of all properties of the classical
fixed point index.

In another direction let \( X \) be a bounded, complete metric space
and \( f: X \rightarrow X \) a \( k \)-set-contraction, \( k < 1 \). Using Darbo's results we can
prove that \( \text{cl}(\cap_{n \geq 1} f^n(X)) \) is nonempty and compact. In general
Browder [3] has suggestively called such maps asymptotically com­
 pact and has proved fixed point theorems about them. Such theorems
have proved useful in studying ordinary differential equations. We
generalize Browder's chief result to the context of \( k \)-set-contractions.

2. The fixed point index for \( k \)-set-contractions. Let us begin by
recalling the basic properties of the classical fixed point index. Let \( X \)}
be a compact, metric absolute neighborhood retract (ANR) and G
an open subset of X. Let \( f: G \rightarrow X \) be a continuous function and as­
sume the fixed point set of \( f \) in G is compact (it may be empty). Then
we can define an integer \( i_X(f, G) \) which has the following properties
(properties all spaces here are compact, metric ANR's):

(a) Let \( I \) denote the closed unit interval \([0, 1]\) and let \( \Omega \) be an open
subset of \( X \times I \). Let \( F: \Omega \rightarrow X \) be a continuous map and assume that
\( \{ (x, t) \in \Omega \mid F(x, t) = x \} \) is a compact subset of \( \Omega \). Let 
\( \Omega_i = \{ (x, t) \in \Omega \mid F(x, t) = x \} \) and \( F_i = F(\cdot, i) \). Then we have
\( i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1) \) (the homotopy
property).

(b) Let \( f: G \rightarrow X \) be a continuous function and assume that \( S = \{ x \in G \mid f(x) = x \} \) is a compact subset of \( G \). Let \( G_1 \) and \( G_2 \) be disjoint
open subsets of \( G \) such that \( S \subseteq G_1 \cup G_2 \). Then we obtain
\( i_X(f, G) = i_X(f, G_1) + i_X(f, G_2) \) (the additivity property).

(c) Let \( f: X \rightarrow X \) be a continuous function and let \( \Lambda(f) \) denote the
Lefschetz number of \( f \). Then we have \( i_X(f, X) = \Lambda(f) \) (the normaliza­
tion property).

(d) Let \( X_1 \) and \( X_2 \) be compact, metric ANR's, \( G_1 \) and \( G_2 \) open sub­
sets of \( X_1 \) and \( X_2 \) respectively. Suppose \( f_1: G_1 \rightarrow X_1 \) and \( f_2: G_2 \rightarrow X_1 \), so
that \( f_2 f_1: f_1^{-1}(G_2) \rightarrow X_1 \) and \( f_1 f_2: f_2^{-1}(G_1) \rightarrow X_2 \). Assume that
\[ S_1 = \{ x \in f_1^{-1}(G_2) \mid (f_2 f_1)(x) = x \} \]
is compact. Then \( S_2 = \{ x \in f_2^{-1}(G_1) \mid (f_2 f_1)(x) = x \} \) is compact and
\( i_X(f_2 f_1, f_1^{-1}(G_2)) = i_X(f_1 f_2, f_2^{-1}(G_1)) \) (the commutativity property).

The four properties listed here are slight variants of the usual
properties proved in the literature for the fixed point index. They
can be proved without too much difficulty.

Let us introduce some notation. Let \( X \) be a closed subset of a
Banach space \( B \). We shall say \( X \in \mathcal{F} \) if we can write \( X = \bigcup_{i=1}^{\infty} C_i \), where
\( C_i \) are closed, convex sets in \( B \). The metric on \( X \) will always be that
which it inherits from \( B \). Actually, the following results hold if we
only know that \( X \) is a locally finite union of closed, convex sets, i.e.
\( X = \bigcup_{\alpha \in A} C_\alpha \) and every \( x \in X \) has a neighborhood \( N_\alpha \) such that
\( N_\alpha \cap C_\alpha = \emptyset \) except for finitely many \( \alpha \).

Let \( G \) be a subset of a Banach space \( B \) and \( f: G \rightarrow B \) a continuous
map. Let us write \( K_1(f, G) = \text{cocl}(f(G)) \), \( K_\alpha(f, G) = \text{cocl}(f(G \cap K_{\alpha-1}(f, G))) \),
and \( K_\alpha(f, G) = \bigcap_{\alpha \geq 1} K_\alpha(f, G) ; \text{cocl} \) denotes convex closure. It is easy
to see that \( f: G \cap K_\alpha(f, G) \rightarrow K_\alpha(f, G) \) and \( K_\alpha(f, G) \) is closed and con­
vex. If \( G \) is bounded and \( f: G \rightarrow X \) is a \( k \)-set-contraction, \( k < 1 \), Darbo's
results also imply that \( K_\alpha(f, G) \) is compact.

Suppose that \( X \in \mathcal{F} \), \( G \) is an open subset of \( X \) and \( f: G \rightarrow X \) is a
continuous map. Assume that \( S = \{ x \in G \mid f(x) = x \} \) is compact. Fi-
nally, assume that $f$ is a local strict-set-contraction. By this we mean that every point $x \in G$ has a neighborhood $N_x$ such that for $D \subseteq N_x$, $\gamma(f(D)) \subseteq k_D \gamma(D)$, $k_D < 1$. Using these assumptions, we can find a bounded open neighborhood $G_1$ of $S$ such that $f: G_1 \to X$ is a $k$-set-contraction, $k < 1$. Let us write $K^*_S = K^*_S(f, G_1) \cap X$; $K^*_S$ is a compact, metric ANR, $G_1 \cap K^*_S$ is an open subset of $K^*_S$, and $f: G_1 \cap K^*_S \to K^*_S$ is a continuous function satisfying the necessary condition, so $i_{K^*_S}(f, G_1 \cap K^*_S)$ is defined. We define $i_X(f, G) = i_{K^*_S}(f, G_1 \cap K^*_S)$. All the usual properties carry through to this setting.

(a) $i_X(f, G)$ does not depend on the particular $G_1$ chosen. Further, if $X$ is a compact metric ANR, $i_X(f, G)$ (the usual definition) equals $i_{K^*_S}(f, G_1 \cap K^*_S)$.

(b) Let $I = [0, 1]$ and let $\Omega$ be an open subset of $X \times I$, $X \in S$. Let $F: \Omega \to X$ be a continuous map and assume that $\{(x, t) | F(x, t) = x\}$ is compact. Assume that $F$ is a local strict-set-contraction in the following sense: Given $(x_0, t_0) \in \Omega$, we can find an open neighborhood $N_{(x_0, t_0)} \subseteq \Omega$ of $(x_0, t_0)$ such that for $D \subseteq X$, $\gamma(F(N_{(x_0, t_0)} \cap (D \times I))) \subseteq k_{(x_0, t_0)} \gamma(D)$, $k_{(x_0, t_0)} < 1$. Then $i_X(F, \Omega)$ is defined for $t \in I$ and $i_X(F, \Omega) = i_X(F, \Omega)$ (the homotopy property).

If $\Omega$ is of the form $G \times I$, where $G$ is a bounded, open set, and $F$ is defined on $\text{cl}(G) \times I$, then if $F(x, t) \neq x$ for $x \in \partial G$ and $\gamma(F(A \times I)) \subseteq k\gamma(A)$, $k < 1$, for $A \subseteq \text{cl}(G)$, the conditions of the homotopy property are met. This latter condition is satisfied if each $F_t: \text{cl}(G) \to X$ is a $k$-set-contraction, $k < 1$, $k$ independent of $t$, and the map $t \to F_t$ is continuous from $I$ to the sup topology for bounded, continuous functions on $G$.

(c) Let $G$ be an open subset of a space $X \in S$ and let $f: G \to X$ be a local strict-set-contraction. Assume $S = \{x \in G | f(x) = x\}$ is compact and $S \subseteq G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint open subsets of $G$. Then we have $i_X(f, G) = i_X(f, G_1) + i_X(f, G_2)$ (the additivity property).

(d) Assume $X \in S$ and let $f: X \to X$ be a $k$-set-contraction, $k < 1$. Suppose that $n^*(X)$ is bounded for some $n$. Then $i_X(f, X)$ and $\Lambda_{\text{gen}}(f)$ are defined and $i_X(f, X) = \Lambda_{\text{gen}}(f)$, where $\Lambda_{\text{gen}}(f)$ denotes the generalized Lefschetz number defined by Leray in [8] (the normalization property).

(e) Let $G_1$ and $G_2$ be open subsets of spaces $X_1$ and $X_2$, $X_i \in S$. Let $f_1: G_1 \to X_1$ and $f_2: G_2 \to X_1$ be, respectively, $k_1$- and $k_2$-set-contractions, $k_1, k_2 < 1$. If $f_1$ is a 0-set-contraction, we only need to assume $f_2$ is continuous. Let $S_1 = \{x \in f_1^{-1}(G_2) | (f_2 f_1)(x) = x\}$ and assume that $S_1$ is compact. It follows then that $S_2 = \{x \in f_2^{-1}(G_1) | (f_1 f_2)(x) = x\}$ is compact and $i_X(f_1 f_2, f_1^{-1}(G_2)) = i_X(f_1, f_2^{-1}(G_1))$ (the commutativity property).
The above properties can be derived from the corresponding results for the classical fixed point index with the aid of the following lemma.

**Lemma 1.** Let \( A = \bigcup_{i=1}^{n} C_i \) be a finite union of compact, convex sets \( C_i \) in a Banach space \( X \). Let \( B = \bigcup_{i=1}^{n} D_i \) be another finite union of compact, convex sets with \( C_i \supseteq D_i, 1 \leq i \leq n \). Let \( 0 \) be an open subset of \( A \) and \( \phi: \text{cl}(0) \rightarrow A \) a continuous map such that \( \phi(x) \neq x \) for \( x \in \text{cl}(0) - 0 \). Assume that \( \phi: 0 \cap B \rightarrow B \) and that \( B \supseteq K_{\omega}(\phi, 0) \cap A \). Then \( i_A(\phi, 0) = i_B(\phi, 0 \cap B) \).

Lemma 1, in turn, is proved with the aid of a purely geometrical result which may have some independent interest.

**Lemma 2.** Let \( A_n = \bigcup_{i=1}^{n} C_i \), where \( m \) is independent of \( n \), \( C_i \) is a compact, convex set in a Banach space \( X \), and \( C_i \supseteq C_{i+1}, 1 \leq i \leq m \). Let \( A_\infty = \bigcap_{n \geq 1} A_n \). Then given \( \delta > 0 \), \( A_\infty \subset A_n \) is a deformation retract of \( A_\infty \) for \( n \geq n(\delta) \) and the deformation retraction \( H_n: A_\infty \times I \rightarrow A_n \) can be chosen so that \( \| H_n(x, t) - x \| < \delta \) for \( (x, t) \in A_\infty \times I \).

3. **Degree theory for \( k \)-set-contractions in Banach space.** Let \( G \) be a bounded open subset of a Banach space \( X \), \( I \) the identity on \( X \), and \( \phi: \text{cl}(G) \rightarrow X \) a \( k \)-set-contraction, \( k < 1 \). Assume that \( (I - \phi)(x) \neq a \) for \( x \in \partial G \). Then we define \( \deg(I - \phi, G, a) = i_{x}(f + a, G) \). As one might suspect, when \( \phi = U + C \), \( U \) a strict contraction, \( C \) compact, this definition agrees with that given by Browder and Nussbaum [5].

As a trivial application of the above apparatus, we find the following simple refinement of Darbo's theorem.

**Theorem.** Let \( G \) be a bounded, closed, convex set with nonempty interior. Assume \( f(\partial G) \subset G \). Then \( f \) has a fixed point.

Since Leray has proved an invariance of domain theorem for mappings of the form \( I - C \), \( C \) compact, one might also hope for such a theorem for maps of the form \( I - f \), \( f \) a local strict-set-contraction. This turns out to be true.

**Theorem (Invariance of Domain).** Let \( G \) be an open subset of a Banach space \( X \) and \( f \) a local strict-set-contraction, \( f: G \rightarrow X \). Assume \( (I - f) \) is one to one. Then \( (I - f)(G) \) is open.

The principal lemma for proving the above theorem is the following result. For the case that \( f \) is a compact map, this is a classical theorem.

**Theorem.** Let \( B \) be a closed ball about the origin in a Banach space \( X \) and \( f: B \rightarrow X \) a \( k \)-set-contraction, \( k < 1 \). Assume that \( f(x) \neq x \) for \( x \in \partial B \) and that \( f(-x) = -f(x) \) for \( x \in \partial B \). Then we have \( \deg(I - f, \text{int } B, 0) \neq 0 \).

We also obtain results in other directions. If \( X \) is a uniformly con-
vex Banach space, $G$ a closed, bounded, convex subset of $X$, and $f: G \to X$ continuous, say that $f$ satisfies condition $CC$ if for any $x \in G$ and $\varepsilon > 0$ we can find a weak neighborhood $N_x$ of $x$ in $G$ such that for $u, v \in N_x$, $\|f(u) - f(v)\| \leq \|u - v\| + \varepsilon$. As an example, let $V$ be a non-expansive map, $C$ a completely continuous map, both defined on $G$. Let $U$ be a nonexpansive map defined on $(V + C)(G)$. Then $U(V + C)$ satisfies condition $CC$. If $f$ satisfies condition $CC$, $f$ is a 1-set-contraction.

**Theorem.** Let $f$, $G$, and $X$ be as above. Assume $f: G \to G$ (or $f: \partial G \to G$ if $G$ has nonempty interior). Then $f$ has a fixed point.

This is a generalization of some results of Browder [4] and Kirk [7].

4. **Asymptotic fixed point theorems for $k$-set-contractions.** In a recent article [3], Browder has proved a slight variant of the following theorem.

**Browder's Theorem.** Let $X$ be a metrizable, locally completely metrizable ANR. Let $f: X \to X$ be a continuous map. We make the following assumptions about $f$:

(a) $\bigcap_{n \geq 1} f^n(G)$ is nonempty and has compact closure in $G$.

(b) $f$ is locally compact.

(c) $\text{cl}(\bigcap_{n \geq 1} f^n(X))$ is homologically trivial in some compact set $K \subseteq X$, while for $z \in X$, $\bigcup_{n \geq 0} f^n(z)$ has compact closure. Then $f$ has a fixed point.

We prove the following result, which can be shown (with some effort) actually to include Browder's theorem.

**Theorem.** Let $G$ be an open subset of a space $X \in \mathbb{F}$. Let $f: G \to G$ be a continuous map. We make the following assumptions about $f$:

(a) $\bigcap_{n \geq 1} f^n(G)$ is nonempty and has compact closure in $G$.

(b) $f$ is a local strict-set-contraction.

(c) There is a compact set $K \supseteq \text{cl}(\bigcap_{n \geq 1} f^n(G))$ such that $\text{cl}(\bigcap_{n \geq 1} f^n(G))$ is homologically trivial in $K$ and such that $\bigcup_{n \geq 0} f^n(K)$ has compact closure in $G$. Then we find $i_X(f, G)$ is defined and nonzero. In particular, $f$ has a fixed point.

The fact that $i_X(f, G) \neq 0$ is interesting methodologically, for it suggests that the proofs of asymptotic fixed point theorems are methods of proving a generalized fixed point index is not zero.

As corollaries of the above theorem we can obtain simpler but more elegant results.

**Theorem.** Let $X \in \mathbb{F}$ and $f: X \to X$ be a $k$-set-contraction, $k < 1$. Assume that $f^n(X)$ is bounded for some $n$. It follows that $\bigcap_{n \geq 1} f^n(X)$ is non-
empty and has compact closure $C_\omega$. Assume that $C_\omega$ is homologically trivial in some compact set $K \supset C_\omega$. Then $i_X(f, X) \neq 0$ and $f$ has a fixed point.

**Theorem.** Let $X$ be a closed, convex subset of a Banach space $B$. Let $f: X \to X$ be a $k$-set-contraction, $k < 1$. Assume that $f^n(X)$ is bounded for some $n$. Then $i_X(f, X) \neq 0$, and $f$ has a fixed point.

This last theorem is a direct generalization of one of the earliest asymptotic fixed point theorems [2]: Let $X$ be a Banach space and $C: X \to X$ a continuous map which is compact on bounded sets. Assume that $C^n(X)$ is bounded. Then $C$ has a fixed point.

**References**


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