A METHOD FOR COMPARING UNIVALENT FUNCTIONS

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1. The Loewner representation. Let \( f \) be any function in the class \( S \) of univalent functions on the unit disk bearing the normalization \( f(0) = 0, f'(0) = 1 \). Then it is known [1], [2], that \( f \) can be represented as

\[
f(z) = \lim_{t \to -\infty} e^t h(z, t),
\]

locally uniformly in \( z \) on \( |z| < 1 \), where \( h(z, \cdot) \) is the solution of Loewner's equation in its general form

\[
\frac{dh}{dt} = -hp(h, t) \quad \text{a.e. in } t \text{ on } [0, \infty)
\]

with the initial values

\[
h(z, 0) = z, \quad |z| < 1.
\]

Here \( p(\cdot, t) \) denotes a suitably chosen one-parameter family of holomorphic functions on the unit disk having positive real part and normalized so that \( p(0, t) = 1 \), whose dependence on \( t \) is Lebesgue measurable on \( [0, \infty) \) whenever the first variable is held fixed, and the solution of (2) is understood in the Carathéodory sense [3].

Conversely, if \( p(\cdot, t) \) denotes any family of functions satisfying the above requirements, then the solution to the foregoing initial-value problem is known to exist and be holomorphic and univalent in \( z \) on the unit disk; the limit in (1) is then also known to exist and to determine a function in class \( S[1], [2] \).

More generally, one can consider the general solution \( h(z, s, t) \) of (2) for \( 0 \leq s \leq t \) with the initial values

\[
h(z, s, s) = z, \quad |z| < 1.
\]

Then, in place of (1), one has

\[
\lim_{t \to -\infty} e^t h(z, s, t) = g(z, s)
\]

locally uniformly in \( z \) for \( |z| < 1 \), where \( e^{-g(z, s)} \) now belongs to \( S \) for all \( s \) in \( [0, \infty) \). The function \( g \) is absolutely continuous in \( s \) and constitutes an integral of Equation (2), for it is easily shown that
2. Comparisons, local and global, in the class $S$. The identity (4) can be used as the basis for a local variational theory within the class $S$, by subjecting the unit disk in the $h$-plane to the infinitesimal transformations of the semigroup of bounded univalent functions, which Loewner has characterized [5]. This results in a variation in the function $f$ for each fixed $t$, and, in e.g., the case of extremal problems for coefficients of functions in class $S$, it leads to a condition on the initial coefficients of the functions $\varphi(\cdot, t)$ that generate extremal mappings that amounts to the Pontryagin maximum principle [4], and is equivalent to Schiffer's characterization of the extremal functions as solutions of his quadratic differential equation [1], [6].

What we wish to report on now is the possibility of making global comparisons between the functions in class $S$, in contrast to the local comparisons mentioned above, by considering in place of (4) the function $G(z, t)$ defined by the equation

$$g(h(z, t), t) = G(z, t) \quad (|z| < 1, \quad 0 \leq t < \infty).$$

Here $h$ is understood to be the solution to the initial-value problem made up of (2) and (3) when $\varphi(\cdot, t)$ is replaced by any other one-parameter family $\varphi(\cdot, t)$ that satisfies the same conditions as $\varphi(\cdot, t)$.

The function $G$ belongs to class $S$ for all $t$ in $[0, \infty)$ and satisfies

$$G(z, 0) = f(z)$$

while (by arguments similar to those already used in [1], [2])

$$\lim_{t \to \infty} G(z, t) = \hat{f}(z),$$

locally uniformly in $z$ on the unit disk, where $\hat{f}$ is the function in $S$ generated by $h$ in the same manner as (1). Moreover, $G$ is absolutely continuous in $t$, locally uniformly in $z$, and its $t$-derivative is given by

$$\partial_t G(z, t) = h(z, t) g'(h(z, t), t) \left[ \varphi(h(z, t), t) - \widehat{\varphi}(h(z, t), t) \right] \text{ a.e.,}$$

where the $'$ denotes the derivative of $g$ w.r.t. its first argument.

This procedure makes it possible to join a given function $f$ in $S$ with any other function $\hat{f}$ in $S$ along an absolutely continuous path in $S$, thereby generalizing the procedure of §1, which constitutes the special case of $f(z) = z$ in the present set-up.
3. Application to extremal problems for coefficients of functions in $S$. If we expand both sides of (8) in power series about the origin, we find that the $n$th coefficient of $\partial_t G$ is given by the expression

$$
\sum_{m=1}^{n-1} B_m(t) [\hat{p}_m(t) - \hat{p}_m(t)] \quad \text{a.e.,}
$$

where the $p_m$ and $\hat{p}_m$ are the $m$th coefficients of $p(\cdot, t)$ and $\hat{p}(\cdot, t)$, resp., and the $B_m$ are certain combinations of the initial coefficients of the functions $g$ and $\hat{h}$ making them absolutely continuous as functions of $t$. Their derivatives involve the $p_m$ and $\hat{p}_m$, and when $p(\cdot, t)$ coincides with $\hat{p}(\cdot, t)$ the resulting expressions reduce to a set of differential equations which are already known in the local theory [1], [6].

To establish the global extremality in $S$ of the real part of the $n$th coefficient of a function $f$ generated by (2), it would clearly be enough to show that the real part of (9) is nonpositive for a.e. $t$ whatever the choice of the $\hat{p}_m$ (so long as they come from functions admissible in the sense of §1), for then the real part of the $n$th coefficient of the function $G$ would be nonincreasing. In view of (6) and (7), the global extremal property would thereby be verified.

In practice, it is desirable to restrict the competing functions $\hat{p}(\cdot, t)$ by placing a limitation on the range of their initial coefficients. In certain cases it can be shown by using the symmetries of the coefficient body $S_n$ for functions in $S$ that this limitation results in no loss of generality.

4. Illustration: the case $n = 3$. We shall verify the extremal property of the third coefficient of the Koebe function $f(z) = z/(1-z)^2$ in the class of functions $f(z) = z + \delta z^2 + \delta z^3 + \cdots$ in $S$ which have $\Re \delta \geq 0$; an analogous result will hold for $f(z) = z/(1+z)^2$ when $\Re \delta \leq 0$. We have $p(h, t) = (1-h)/(1+h)$ and $g(h, t) = e^{ih}/(1-h)^2$; if we put $\hat{p}(h, t) = 1 + 2 \sum_{m=1}^{\infty} \hat{p}_m(t) h^m$ and $\hat{h}(z, t) = e^{-z} [z + \sum_{m=2}^{\infty} \hat{b}_m(t) z^m]$ then the real part of (9) for $n = 3$ becomes twice

$$
\text{Re} \left\{ B_1(t) [-1 - \hat{p}_3(t)] + B_2(t) [1 - \hat{p}_2(t)] \right\},
$$

where

$$
B_1(t) = 2\hat{b}_3(t) e^{-t} + 4e^{-2t}, \quad B_2(t) = e^{-2t}.
$$

We compute from (2)

$$
\frac{db_3(t)}{dt} = -2\hat{p}_1(t) e^{-t} \text{ a.e.},
$$

$$
\text{Re} \frac{db_3(t)}{dt} = \text{Re} \left\{ -4\hat{b}_3(t) \hat{p}_1(t) e^{-t} - 2\hat{p}_3(t) e^{-2t} \right\} \text{ a.e.,}
$$
and therefore

\[ \frac{dB_1(t)}{dt} = -B_1(t) + 4e^{-st}[-1 - \dot{b}_1(t)] \quad \text{a.e.,} \]

while from (11) we deduce that \( B_1(0) = 4 \).

In order to make use of the assumption that \( \Re \alpha_2 \geq 0 \), we prove the following lemma.

**Lemma.** If \( \alpha_2 = \lim_{t \rightarrow \infty} b_2(t) \) has nonnegative real part, then the point \( (\alpha_2, \Re \alpha_3) \) can be reached by a solution of (12) that starts at the origin when \( t = 0 \) and has \( \Re \dot{b}_1(t) \leq 0 \) a.e. in \([0, \infty)\).

**Proof.** In view of (12), this is the same as saying that \( \Re b_2(t) \) can be assumed to be monotone nondecreasing. Suppose it is not. Then there will be two values \( \xi < \eta \) of \( t \) such that \( \Re b_2(\xi) = \Re b_2(\eta) \). On the interval \([\xi, \eta]\) we can, if need be, replace \( \dot{p}(\cdot, t) \) by the functions (also of positive real part)

\[
q(h, t) = \frac{1}{2}[\dot{p}(h, t) + (\dot{p}(-h, t))^-] = 1 + 2i \Im \dot{b}_1(t)h + 2 \Re \dot{b}_1(t)h^2 + \cdots.
\]

This makes \( \Re b_2(t) \) constant on \([\xi, \eta]\) and leaves \( \Im b_2(t) \) unchanged, while in the equation for \( \Re d b_2(t)/dt \) the only change is that the term

\[ -4 \Re b_2(t) \Re \dot{b}_1(t)e^{-t} \]

is now missing. But

\[
\int_{\xi}^{\eta} -4 \Re b_2(t) \Re \dot{b}_1(t)e^{-t} dt = [\Re b_2(\eta)]^2 - [\Re b_2(\xi)]^2 = 0,
\]

so that the missing term does not affect the value of \( \Re b_2(\eta) \). By the Rising Sun Lemma, there are at most a countable number of disjoint intervals in \([0, \infty)\) where this alteration of \( \dot{p}(\cdot, t) \) needs to be made, so that the altered \( \dot{p}(\cdot, t) \) remains measurable in \( t \) and yields a trajectory of (12) that satisfies the assertion of the lemma.

A similar reasoning, in which \( \dot{p}(h, t) \) is replaced by

\[
q(h, t) = \frac{1}{2}[\dot{p}(h, t) + (\dot{p}(-h, t))^-],
\]

shows that we can also assume that \( \Im \dot{b}_1(t) \) does not change sign on \([0, \infty)\). For \( \Re \dot{b}_1(t) \) and \( \Im \dot{b}_1(t) \) restricted in this way, Equation (13) and the initial condition \( B_1(0) = 4 \) imply that

\[ \Im B_1(t) \Im \dot{b}_1(t) \leq 0 \quad \text{a.e.,} \]
and

$$4e^{-2t} \leq \text{Re} B_1(t) \leq 4e^{-t}$$

(since now $0 \leq 1 + \text{Re} \, \hat{p}_1(t) \leq 1$ a.e.).

To prove that (10) is nonpositive it is therefore enough to show that

$$-4[1 + \text{Re} \, \hat{p}_1(t)] \leq \text{Re} \, \hat{p}_2(t) - 1 \text{ a.e.}$$

We may restrict $\hat{p}_1(t)$ and $\hat{p}_2(t)$ to the form $\exp(i\theta(t))$, $\exp(2i\theta(t))$, resp., for $\theta(t)$ real-valued, either by appealing to Loewner's theory of slit mappings [5] or, even better, to the Carathéodory representation of $\hat{p}_1(t)$ and $\hat{p}_2(t)$ [1], [7]. Then (14) is equivalent to the inequality $-2[1 + \cos \theta(t)]^2 \leq 0$ a.e., and the monotonicity of the real part of the third coefficient of $G$ is thereby proved.

This gives us Loewner's inequality $\text{Re} \, a_3 \leq 3$ when $\text{Re} \, a_2 \geq 0$, and at the same time shows that equality holds only when $\hat{f}$ is the Koebe function $z/(1-z)^2$. (If one inspects the real part of the second coefficient of $G$, it also is seen to be monotone decreasing, so the same Koebe function is extremal there, too.)

By a variant of the foregoing procedure one can prove a number of other inequalities, among which is Jenkins' inequality

$$\text{Re} \left[ e^{i\phi}(a_4 - a_2^2) - \lambda e^{i\phi}a_2 \right] \leq 1 + 3\lambda^2/8 + (\lambda^2 \log 4/\lambda)/4$$

for $\phi$ real and $0 < \lambda \leq 4$ [8].

References


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