1. The purpose of this note is to establish some nonzero elements in the homotopy groups of spheres. This results from unstabilizing a method of Adams. Namely, an Adams spectral sequence is used to detect elements in $\pi_{n+i}(S^n)$ for various $n$ and $i$; in addition to the $d$ and $e$ invariants of Adams, the Hopf invariants are used to show that certain of these elements are nonzero. One consequence will be the following.

**Consequence.** The groups $\pi_{4+i}(S^4)$ are nonzero for all $i \geq 0$.

2. Recall the mod-$p$-restricted lower central series spectral sequence (abbr: mod-$p$-RLCSSS), constructed as in [4], [5] and [10]. For each simplicial set $X$, form $GX$ as in [6], filter $GX$ by its mod-$p$-RLCS, and pass to the homotopy exact couple. The resulting spectral sequence we will label $E_{s,d}(X)$, where $s =$ filtration and $d =$ dimension. The results of [4, §(2.4)] show that for the sphere spectrum $S$, the term $E^1(S)$ of the mod-$2$-RLCSS is a ring $A^*$ with multiplicative generators $\lambda_i$ for each $i \geq 0$. An additive basis for $E^1(S)$ consists of all monomials $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k}$, where $I = (i_1, \cdots, i_k)$ is a sequence of nonnegative integers with $2i_j \leq i_{j+1}$ for $j = 1, 2, \cdots, k-1$. Call such monomials allowable. In the unstable case, the results of [4, §(5.4)] show that for the $n$-sphere $S^n$, $E^1(S^n)$ is the subvector space of $A^*$ with basis all $\lambda_I$ which are allowable and for which $i_1 < n$. Such a monomial $\lambda_I \in E^1(S^n)$, where $I = (i_1, \cdots, i_k)$, has filtration $k$, and dimension $n + \sum i_j$.

3. There is a short exact sequence of differential vector spaces:

$$0 \rightarrow E_{z,n+i}(S^n) \xrightarrow{i} E_{z,n+i+1}(S^{n+1}) \xrightarrow{h} E_{z-1,n+i+1}(S^{2n+1}) \rightarrow 0$$

where $i$ is the inclusion and $h$ is defined on the allowable basis by

$$h(\lambda_I \lambda_t) = \lambda_I \quad \text{for } j = n,$$

$$= 0 \quad \text{for } j < n.$$  

From this, there derives a long exact sequence

$$(3.1) \quad \cdots \rightarrow E^0(S^n) \xrightarrow{i_*} E^0(S^{n+1}) \xrightarrow{h_*} E^0(S^{2n+1}) \rightarrow \cdots.$$  

It can be shown that $h_*$ commutes with all differentials, and is induced
by the Hopf-invariant in the SHP-sequence of Whitehead, James:

\[ \cdots \to \pi_{n+1}(X^n) \to \pi_{n+i+1}(S^{n+1}) \to \pi_{n+i+1}(S^{2n+1}) \to \cdots. \]

From the sequence (3.1), some calculations in \( E^2(S^n) \) can easily be made.

4. For each \( m \geq 0 \), define functions \( \phi_2(m), \phi_3(m), \phi_4(m), \phi_5(m), \phi(m) \) by the rules:

\[
\begin{array}{c|cccccccc}
 m = 8k+ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\phi_2(m) = 4k+ & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\
\phi_3(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\
\phi_4(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\
\phi_5(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
\phi(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
\end{array}
\]

The function \( \phi(m) \) describes the Adams vanishing line: \( \text{Ext}^s_{\mathbb{A}}(\mathbb{Z}_2, \mathbb{Z}_2) = 0 \) for \( s > \phi(i-s) \). Unstably, the functions \( \phi_n(m) \) (set \( \phi_n(m) = \phi(m) \) for \( n \geq 6 \)) also describe a vanishing line, possibly modulo a tower, as follows.

**Theorem.** \( E^2_{i+n+1}(S^n) = 0 \) for \( s > \phi_n(i) \), except for the tower at \( i = 0 \), and the tower which occurs when \( n \) is even and \( i = n-1 \).

This can be proven using the stable vanishing line \( \phi(m) \) of Adams [1], (3.1), and downward induction.

**Corollary.** In the 2-component of \( \pi_{n+i}(S^n) \), each element has order \( \leq 2^{\phi_n(i)} \).

This is of course the unstable analogue of [1, p. 69]. There is also a similar vanishing line for each prime \( p \), and all together give a bound for the order of any element (of finite order).

5. Let \( P \) be the periodicity operator defined by the Massey product \( P(x) = \{ x, \lambda_0, \lambda_7 \} \). The following table describes some (not all) non-zero elements in \( E^2(S^n) \) near the vanishing line. They are cycles in every \( E^r(S^n) \) for which they are defined, as the differentials on them land in the vanishing-zone or in a tower.
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TABLE

<table>
<thead>
<tr>
<th>Stem dim ( i )</th>
<th>Filtration ( s )</th>
<th>Minimum value of ( n )</th>
<th>Element in ( E^2(S^n) )</th>
<th>Stable element in ( \text{Ext}(Z_2, Z_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 8k )</td>
<td>( 4k - 1 )</td>
<td>3</td>
<td>( P^{k-1}(\lambda_3 \lambda_5^2) )</td>
<td>( P^{k-1}(c_0) )</td>
</tr>
<tr>
<td>( 8k + 1 )</td>
<td>( 4k )</td>
<td>2</td>
<td>( P^{k-1}(\lambda_3 \lambda_5^3) )</td>
<td>( P^{k-1}(h_1c_0) )</td>
</tr>
<tr>
<td>( 8k + 2 )</td>
<td>( 4k + 1 )</td>
<td>3</td>
<td>( P^k(\lambda_1) )</td>
<td>( P^k(h_1) )</td>
</tr>
<tr>
<td>( 8k + 3 )</td>
<td>( 4k + 2 )</td>
<td>2</td>
<td>( P^k(\lambda_1^2) )</td>
<td>( P^k(h_1^2) )</td>
</tr>
<tr>
<td>( 8k + 4 )</td>
<td>( 4k + 3 )</td>
<td>5</td>
<td>( P^k(\lambda_1^3) )</td>
<td>( P^k(\lambda_1 h_1) )</td>
</tr>
<tr>
<td>( 8k + 5 )</td>
<td>( 4k + 4 )</td>
<td>2</td>
<td>( P^k(\lambda_1^4) )</td>
<td>( P^k(\lambda_1^2 h_1) )</td>
</tr>
<tr>
<td>( 8k + 6 )</td>
<td>( 4k + 5 )</td>
<td>4</td>
<td>( P^k(\lambda_1^5) )</td>
<td>( P^k(\lambda_1^3 h_1) )</td>
</tr>
<tr>
<td>( 8k + 7 )</td>
<td>( 4k + 6 )</td>
<td>5</td>
<td>( P^k(\lambda_1^6) )</td>
<td>( P^k(\lambda_1^4 h_1) )</td>
</tr>
</tbody>
</table>

The elements \( P^{k-1}(c_0) \), \( P^{k-1}(h_1c_0) \), \( P^k(h_1) \), \( P^k(h_2) \), \( P^k(h_1h_2) \), \( P^k(h_2^2h_3) \), \( P^k(h_2^3h_4) \) are shown never to be boundaries in the stable Adams spectral sequence because of nonzero \( d \) or \( e \) invariants; see [2], [7], [8], [9]. Hence, by naturality of suspension, their precursors are never boundaries in each \( E^r(S^n) \) of the mod-2-RLCSSS.

The Hopf-invariant \( h_*: E^r(S^5) \to E^r(S^5) \) shows that the elements \( P^k(\lambda_3 \lambda_5^2) \), \( P^k(\lambda_5^3) \) are not boundaries in any \( E^r(S^n) \), since \( h_* \) of them are not boundaries in \( E^r(S^n) \). Similarly, the elements \( P^k(\lambda_3 \lambda_1^2) \), \( P^k(\lambda_3 \lambda_1^3) \) and \( P^k(\lambda_3 \lambda_1^4) \) are never boundaries in any \( E^r(S^n) \).

6. For odd primes \( p \), the \( E^1 \)-term of the mod-\( p \)-RLCSSS for odd spheres is described in [4, §8]. The analogous vanishing statement is \( E^2_{s, s+1}(S^n) = 0 \), for all odd \( n \), and \( s > [i+3/2p-2] \). Also, in filtration \( k \) and dimension \( 3+2k(p-1) - 1 \), \( E^2(S^3) \) has a single generator say \( a_k \). As all differentials on \( a_k \) land in the vanishing zone, \( a_k \) is a permanent cycle; also, \( a_k \) is never a boundary, shown by a mod-\( p \) version of [9]. Thus \( a_k \) detects a nonzero class of order \( p \) in \( \pi_{i+2k(p-1)-1}(S^3) \). Of course the element detected by \( a_k \) is just (a nonzero multiple of) Toda's \( x_k \) shown to be nonzero by Adams' \( e \)-invariant argument.

7. It is now easy to exhibit some nonzero homotopy classes, as each of the elements in the table detects a nonzero class in \( \pi_{\ast}(S^n) \) for the corresponding value of \( n \). Using also the elements \( x_k(3) \) for stems
\( \equiv 7 \pmod{8} \), there follows consequence (1). Further, \( \pi_{a+i}(S^a) \) is non-zero at least for all \( i \not\equiv 6 \pmod{8} \), and hence also \( \pi_{a+i}(S^a) \) is nonzero at least for all \( i \not\equiv 7 \pmod{8} \).

**BIBLIOGRAPHY**


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