MAXIMAL ABELIAN SUBALGEBRAS IN HYPERFINITE FACTORS

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1. Introduction. In this note we outline a construction which, together with a new invariant, gives still more maximal abelian subalgebras of the hyperfinite factor of type II₁. We also state some results concerning maximal abelian subalgebras of hyperfinite factors of type III. Complete proofs will appear elsewhere.

First we will establish some notation and terminology. Let $\mathcal{A}$ and $\mathcal{B}$ be maximal abelian subalgebras of a factor $\mathfrak{M}$. We call $\mathcal{A}$ and $\mathcal{B}$ equivalent (in $\mathfrak{M}$) if there is an automorphism of $\mathfrak{M}$ carrying $\mathcal{A}$ onto $\mathcal{B}$. Let $N(\mathcal{A}) = N^1(\mathcal{A})$ be the subalgebra of $\mathfrak{M}$ generated by all those unitary operators $U$ in $\mathcal{A}$ with $U^* U = \mathcal{A}$, and let $N^j(\mathcal{A}) = N(N^{j-1}(\mathcal{A}))$ for $j > 1$. Following Dixmier [2] and Anastasio [1], we call $\mathcal{A}$ regular if $N(\mathcal{A}) = \mathcal{A}$, semiregular if $N(\mathcal{A})$ is a factor distinct from $\mathfrak{M}$, and $n$-semiregular ($n \geq 2$) if no $N^j(\mathcal{A})$, $1 \leq j < n$ is a factor but $N^n(\mathcal{A})$ is. We say that $\mathcal{A}$ has proper [improper] length $n$ if $n$ is the smallest positive integer such that $N^j(\mathcal{A}) = N^{j+1}(\mathcal{A})$ and if $N^n(\mathcal{A}) = \mathcal{A}$ [$N^n(\mathcal{A}) \neq \mathcal{A}$]. Notice that the length of a maximal abelian subalgebra is invariant under equivalence. We are now able to state our main results.

Theorem 1. For each choice of $n = 2, 3$ and $k = 0, 1, 2, \cdots$, the hyperfinite $\text{II}_1$ factor contains an $n$-semiregular maximal abelian subalgebra of improper length $n+k$.

Theorem 2. Let $\mathfrak{M}$ be one of the hyperfinite type $\text{III}_1$ factors of Powers (cf. [5]). Then $\mathfrak{M}$ contains a regular and two inequivalent semiregular maximal abelian subalgebras. Also, for each choice of $n = 2, 3$ and $k = 0, 1, 2, \cdots$, $\mathfrak{M}$ contains two $n$-semiregular maximal abelian subalgebras, one of proper length $n+k$ and one of improper length $n+k$.

Remarks. (1) These results are a summary of the author's doctoral thesis written at the University of British Columbia under the supervision of Dr. D. Bures.

(2) Anastasio has shown that Theorem 1 holds with "improper" replaced by "proper" [1]. Because of a previous remark, our subalgebras are mutually inequivalent as well as inequivalent to those of Anastasio.

(3) $\mathcal{A}$ has proper length $n$ if and only if $\mathcal{A}$ has length $n-1$ in the sense of Tauer [7].
(4) Pukánszky has given a general method for constructing maximal abelian subalgebras in a wide class of type III factors [6]; but, because of an error in the proof of Lemma 17, the types of these subalgebras is not known.

2. Construction of II₁ factors. Let $G$ be a countably infinite group with identity $e$ and let $\Delta$ be the set of all functions with finite support from $G$ into $\{0, 1\}$. Under point-wise addition modulo 2, $\Delta$ becomes an abelian group. For $g \in G$ and $\alpha \in \Delta$, define elements $g\alpha$ and $\alpha g$ in $\Delta$ by

$$
(g\alpha)(h) = \alpha(g^{-1}h) \quad (h \in G),
$$

$$
(\alpha g)(h) = (\alpha(g))^2 - 2\alpha(g) + 1 \quad h = g,
$$

$$
= \alpha(h) \quad \text{otherwise}.
$$

Let $H$ be a Hilbert space with orthonormal basis $(\phi_\alpha)_{\alpha \in \Delta}$. For each $g \in G$, define operators $F_g$ and $U_g$ on $H$ by

$$
F_g\phi_\alpha = \frac{1}{2}\phi_\alpha + \frac{1}{2}\phi_{\alpha g} \quad (\alpha \in \Delta),
$$

$$
U_g\phi_\alpha = \phi_{g\alpha} \quad (\alpha \in \Delta).
$$

Notice that $\{F_g: g \in G\}$ is a commuting family of projections, that $g \to U_g$ is a unitary representation of $G$ on $H$, and that $U_gF_hU_g^* = F_{gh}$ for all $g, h \in G$. It is easy to verify that $\mathfrak{M}$, the von Neumann algebra on $H$ generated by $\{F_g: g \in G\}$, is maximal abelian in $\mathcal{L}(H)$ (use [3, p. 109, Exercise 5] and $\phi_0$). Let $\mathfrak{B}(G)$ be the von Neumann algebra on $H \otimes K$ generated by

$$
\{(M \otimes I)(U_g \otimes V_g): M \in \mathfrak{M}, g \in G\},
$$

where $K$ is the Hilbert space of all complex-valued square-summable functions on $G$ and $V_g$ is the unitary operator on $K$ satisfying $(V_gx)(h) = x(g^{-1}h)$ for all $h \in G$, $x \in K$. Using some standard results from [3, pp. 127–137], one proves

**Lemma 3.** $\mathfrak{B}(G)$ is a factor of type II₁ which is hyperfinite whenever $G$ is the increasing union of a sequence of finite subgroups.

3. Subalgebras of II₁ factors. For a subgroup $G_0$ of $G$, let $N(G_0)$ be the normalizer of $G_0$ in $G$ and let

$$
\mathfrak{M}(G_0) = \mathfrak{R}(U_g \otimes V_g: g \in G_0).
$$

Notice that $\mathfrak{M}(G_0)$ is always a proper subalgebra of $\mathfrak{B}(G)$. In the following three lemmas, $G_c$ denotes a subgroup of $G$. 
LEMMA 4 (Cf. [6, Lemma 14]). \( \mathfrak{M}(G_0) \) is maximal abelian in \( \mathfrak{B}(G) \) if \( \{ghg^{-1} : g \in G_0\} \) is infinite whenever \( h \in G - G_0 \).

PROOF. A straightforward calculation.

LEMMA 5. \( \mathfrak{M}(G_0) \) is a factor if and only if all nontrivial conjugate classes of \( G_0 \) are infinite.

PROOF. Observe that \( \mathfrak{M}(G_0) \) is *-isomorphic to the group operator algebra over \( G_0 \), and apply [4, Lemma 5.3.4].

LEMMA 6 (Cf. [6, Lemma 17]). Suppose that \( G_0 \) satisfies: given a finite subset \( F \subset G \) and a \( g \in G \), there are infinitely many elements \( g_0 \in G_0 \) such that

(i) \( h, k \in F \) and \( hg_0k^{-1} = g_0 \) imply \( h = k \),

(ii) if \( g \in N(G_0) \), then also \( gg_0g^{-1} \in G_0 \). Then \( N(\mathfrak{M}(G_0)) = \mathfrak{M}(N(G_0)) \).

4. Proofs of the theorems. Applying the four preceding lemmas to the groups and subgroups used to prove Theorems I and II of [1], our Theorem 1 follows. The proof of Theorem 2, which is very involved technically, is based on the proof of [6, Lemma 17].

REFERENCES


