APPLICATIONS OF AFFINE ROOT SYSTEMS TO
THE THEORY OF SYMMETRIC SPACES

BY LAWRENCE CONLON

Communicated by Raoul Bott, November 25, 1968

Introduction. Let \((G; K_1, K_2)\) be a compact symmetric triad in the
sense of [3], \(G\) simply connected. The natural action of \(K_1\) on \(G/K_2\)
is of interest because it is variationally complete [5]. In [3] we intro­
duced certain “affine root systems” in order to describe the orbits of
this \(K_1\)-action, and in the present note we wish to announce the classi­
fication [4] of these systems and to indicate further applications to
the theory of symmetric spaces.

1. Preliminaries. Let \(g\) be a complex semisimple Lie algebra, \(\nu\) an
automorphism of \(g\), and set \(g_\nu = \{X \in g; \nu(X) = X\}\). The following is
due essentially to de Siebenthal [7] (cf. also [4, §7]).

\[\begin{equation}
\text{(1.1) Proposition.} \quad \text{If} \ h_0 \subset g, \ \text{is a Cartan subalgebra, there is a unique}
\text{Cartan subalgebra} \ \hat{h} \subset g \ \text{such that} \ h_0 \subset \hat{h}. \ \text{There is a finite family} \ \alpha = \{\xi; \ h_0 \to \mathbb{C}/i\mathbb{Z}\}
\text{of affine functionals and an orthogonal direct sum}
\text{decomposition} \ \ g = \mathfrak{h} \oplus \sum_{\xi \in \alpha} \mathfrak{g}_\xi, \quad \xi \in \alpha
\end{equation}\]

where \(\dim(\mathfrak{h}) = 1\) and

\[\nu \circ \exp(\text{ad}(Z)) \mid g_\xi = \exp(2\pi i \xi(Z)),\]

for all \(Z \in \mathfrak{h}_0\) and \(\xi \in \alpha\). \(\xi(0)\) is pure imaginary for all \(\xi \in \alpha\).

\[\begin{align*}
\mathfrak{h}_0 &= V \oplus iV \text{ where } V \text{ is the real subspace on which the “linear parts”}
\omega = \omega - \omega(0) \text{ of the elements } \omega \in \alpha \text{ are real. One defines}
\mathfrak{A} &= \{\omega \mid V - i\omega(0); \omega \in \alpha\}
\end{align*}\]

interpreted as a set of affine functionals \(V \to \mathbb{R}/\mathbb{Z}\). This is the system
defined by de Siebenthal.

\[g = g_0 \oplus i g_0\] where \(g_0\) is the compact real form of \(g\). Let \(s_1\) and \(s_2\)
be involutive automorphisms of \(g_0\), \(s_1\) and \(s_2\) the extensions of these to
anti-involutions of \(g\). There correspond symmetric subalgebras \(\mathfrak{h}_1, \mathfrak{h}_2\) of
\(g_0\) and noncompact real forms \(g_1, g_2\) of \(g\).

Let \(m \subset g_0\) be the simultaneous \(-1\) eigenspace of \(s_1\) and \(s_2\). Set
\(\nu = s_1 s_2\) and choose \(h_1\), as in (1.1), but such that \(h_1 \cap (m \oplus i m)\) is maxi-

---

1 Research partially supported by NSF GP-6330.

610
mal abelian in \( m \oplus im \). Let \( \sigma \) denote \( \sigma |_g = \sigma |_g \). Note that \( \sigma(V) = V \) and that \( \sigma \) induces a permutation \( \sigma_* \) of \( \mathfrak{A} \). The pair \( (\mathfrak{A}, \sigma_*) \) will be called the affine \( \sigma \)-system associated to \( (g; g_1, g_2) \) (or to \( (g*; \mathfrak{i}_1, \mathfrak{i}_2) \)).

If we let \( V^- \) denote the \(+1\) eigenspace of \( \sigma | V \) and \( \mathfrak{A}^- \) the set of nonconstant restrictions of elements of \( \mathfrak{A} \) to \( V^- \), we obtain the affine root system of \( \mathfrak{A} \).

2. Equivalences and classification. One defines isomorphism \( (\mathfrak{A}, \sigma_*) \cong (\mathfrak{A}', \sigma'_*) \) via linear isometries \( \phi: V \to V' \) carrying \( \mathfrak{A}' \to \mathfrak{A} \) and such that \( \phi \circ \sigma = \sigma' \circ \phi \), and one similarly defines affine equivalence \( (\mathfrak{A}, \sigma_*) \sim (\mathfrak{A}', \sigma'_*) \) via affine isometries \( \phi: V \to V' \) with \( \phi \circ \sigma = \sigma' \circ \phi \).

Isomorphism \( (g; g_1, g_2) \cong (g' ; g'_1, g'_2) \) is defined via an automorphism \( \theta \) of \( g \) leaving \( g_* \) invariant such that \( \theta(g_i) = g'_i \), \( i = 1, 2 \). Affine equivalence \( (g; g_1, g_2) \sim (g' ; g'_1, g'_2) \) means that there are inner automorphisms \( \mathfrak{i}_1, \mathfrak{i}_2 \) of \( g \) leaving \( g_* \) invariant such that \( (g; g_1, g_2) \cong (g; \mathfrak{i}_1(g'_1), g'_2) \).

(2.1) Theorem. Let \( (g; g_1, g_2) \) and \( (g' ; g'_1, g'_2) \) have respective affine \( \sigma \)-systems \( (\mathfrak{A}, \sigma_*) \) and \( (\mathfrak{A}', \sigma'_*) \). Then \( (g; g_1, g_2) \cong (g' ; g'_1, g'_2) \Rightarrow (\mathfrak{A}, \sigma_*) \cong (\mathfrak{A}', \sigma'_*) \Rightarrow (\mathfrak{A}, \sigma_*) \sim (\mathfrak{A}', \sigma'_*) \Rightarrow (g; g_1, g_2) \sim (g' ; g'_1, g'_2) \) for a suitable permutation \( \mathfrak{w} \) of \( \{1, 2\} \). Likewise, \( (g; g_1, g_2) \sim (g' ; g'_1, g'_2) \Rightarrow (\mathfrak{A}, \sigma_*) \sim (\mathfrak{A}', \sigma'_*) \Rightarrow (g; g_1, g_2) \sim (g' ; g'_1, g'_2) \).

The affine \( \sigma \)-systems for all triads \( (g; g_1, g_2) \) have been classified up to affine equivalence [3].

3. Topological applications. Consider the action of \( K_1 \) on \( G/K_2 \) as in the introduction. Let \( T \subseteq G/K_2 \) be the flat geodesic torus described in [3] and [6]. Then \( T \) meets orthogonally every \( K_1 \)-orbit and \( V^- \) identifies in a natural way with the universal covering of \( T \). The system \( \mathfrak{A}^- \) describes the singular set in \( T \) relative to the \( K_1 \)-action [3] and enables us to apply the theory of [2]. If \( N \subseteq G/K_2 \) is a \( K_1 \)-orbit, Theorem 3.1 of [3] shows that the space \( \Omega(G/K_2; x, N) \) of paths on \( G/K_2 \) from the point \( x \) to the submanifold \( N \) has no torsion in homology iff a certain "regularity" condition [3, p. 236] is satisfied by \( \mathfrak{A}^- \). As a result of [4] we can list up to affine equivalence (and a permutation of \( \{1, 2\} \)) the triads \( (g*; \mathfrak{i}_1, \mathfrak{i}_2) \) for which \( \mathfrak{A}^- \) is regular.

For \( g_* \) simple these are given in the following list.

**Type A.** \((A_r; A_q \times A_{r-1} \times R, A_4 \times A_{r-1} \times R), (A_{2r-1}; D_r, A_{2r-4} \times R), (A_{2r}; B_r, A_{2r-5} \times R), (A_{2r-1}; C_r, C_r), (A_{2r-1}; C_r, D_r), (A_{2r-1}; C_r, A_4 \times A_{2r-4} \times R), (A_{2r-1}; D_r, A_4 \times A_{2r-4} \times R)\).

**Type B.** \((B_r; D_r, D_r), (B_r; D_r, B_r)\).

**Type C.** \((C_r; C_q \times C_{r-9}, C_q \times C_{r-5}), (C_r; C_q \times C_{r-9}, A_{r-4} \times R)\).

**Type D.** \((D_r; B_{r-1}, B_{r-1}), (D_r; A_{r-1} \times R, A_{r-1} \times R), (D_r; D_{r-1} \times R, D_k \times D_{r-k}) \) where \( r > k \geq 1 \), \((D_{2r+k}; D_r \times D_{r+k}, A_{r-1} \times R) \) where \( k \geq 0 \),
(Dr; B_{r-1}, D_r \times D_{r-1})$ where $r > k \geq 1$, $(Dr; A_{r-1} \times R, B_r \times B_{r-k-1})$ where $r > k \geq 1$, $(D_4; B_3, \omega(B_4))$, $(D_4; B_3, \omega(B_1 \times B_3))$. Here $\omega$ is the triality automorphism of $D_4$; $B_3$ and $B_1 \times B_3$ are standardly imbedded in $D_4$.

Type E. $(E_6; D_4 \times R, D_4 \times R)$, $(E_6; F_4, F_4)$, $(E_6; D_4 \times R, A_6 \times A_1)$, $(E_6; F_4, D_4 \times R)$, $(E_6; F_4, A_6 \times A_1)$, $(E_7; E_6 \times R, E_6 \times R)$, $(E_7; A_7, E_6 \times R)$, $(E_7; E_6 \times R, D_6 \times A_1)$.

Type F. $(F_4; B_4, B_4)$, $(F_4; B_4, C_3 \times A_1)$.

4. Commuting involutions. Following Hermann [6] one asks whether there is an inner automorphism $\xi$ of $g$ leaving $g_*$ invariant such that $\xi \sigma_2^{-1}$ commutes with $\sigma_2$. Using (1.1) and (2.1) one can prove the answer is affirmative iff $(\mathfrak{g}, \sigma_2) \sim (\mathfrak{h}', \sigma_2')$ where $\phi \in \mathfrak{h}'$ implies $\phi(0) = 0$ or $\frac{1}{2}$.

As Hermann has shown [6, Proposition 2.1], the existence of totally geodesic $K_1$-orbits in $G/K_2$ is completely bound up with the solutions $\xi$ to this problem. The system $(\mathfrak{h}, \sigma_2)$ somewhat clarifies this situation as we now indicate.

Let $\mathfrak{h} = V \rightarrow T$ be the natural covering map. Supposing that the commuting involutions problem has a solution, we lose no generality in assuming $\sigma_2 \sigma_2 = \sigma_2 \sigma_1$ (hence $s_1 s_2 = s_2 s_1$). Then if $\Lambda$ is the lattice $\{X \in V^- : \phi(X) = 0 \text{ or } \frac{1}{2}, \text{ all } \phi \in \mathfrak{h} \}$, we have the following.

(4.1) Proposition. $\Sigma = \mathfrak{h}(\Lambda)$ is the subset of $T$ consisting of the points whose $K_1$-orbits are totally geodesic in $G/K_2$.

The assumption $s_1 s_2 = s_2 s_1$ implies that $s_1$ defines an involutive isometry (again called $s_1$) of $G/K_2$. This situation is quite general.

(4.2) Proposition. Let $G$ be simply connected. Then every involutive isometry of $G/K_2$ having nonempty fixed point set is conjugate (in the isometry group) to one produced by an involutive automorphism $s_1$ of $G$ commuting with $s_2$.

We explicitly identify the fixed point set of the involution $s_1$ in $G/K_2$. For each $\phi \in \mathfrak{h}^-$, let $\phi$ be the linear part as in §1 and define $h_\phi \in V^-$ by $h_\phi \perp \text{Ker}(\phi)$ and $\phi(h_\phi) = 2$. The lattice $\Lambda_\phi$ spanned by these vectors $h_\phi$ is exactly $p^{-1}(\{K_2\})$.

(4.3) Theorem. Again assume $G$ simply connected and $s_1 s_2 = s_2 s_1$. Let $\Lambda_\phi = \frac{1}{2} \Lambda_\phi$ and $\Sigma_\phi = p(\Lambda_\phi)$. Then $\Sigma_\phi \subset \Sigma$ and the fixed point set of $s_1$ in $G/K_2$ is exactly the union of the $K_1$-orbits of the elements of $\Sigma_\phi$.

5. Pseudo-Riemannian symmetric spaces. The explicit solutions of the commuting involutions problem make possible a classification of the isomorphism classes of those $(g; \mathfrak{g}_1, \mathfrak{g}_2)$ for which $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$. For
each of these \((\mathfrak{g}_1, \mathfrak{g}_1 \cap \mathfrak{g}_2)\) and \((\mathfrak{g}_2, \mathfrak{g}_1 \cap \mathfrak{g}_2)\) are dual pseudo-Riemannian symmetric pairs [1]. All pseudo-Riemannian pairs may be obtained in this way; hence [4] contains implicitly the classification [1].

In the following, \(\mathcal{R} = \{ \phi \in \mathfrak{X}: \phi(0) = 0 \}\) and \(\mathcal{R}^- = \{ \phi \in \mathfrak{X}^-: \phi(0) = 0 \}\).

These are identified as subsets of the dual spaces \(V^*\) and \((V^-)^*\) respectively. For other terminology in the theorem below, cf. [1].

\[(5.1) \text{THEOREM.} \text{Let } \mathfrak{g} \text{ be simple, } \sigma_1 \sigma_2 = \sigma_2 \sigma_1. \text{ The corresponding dual symmetric pairs are either both reducible or both irreducible. They are reducible iff } \mathcal{R}^- \text{ spans a subspace of } (V^-)^* \text{ of codimension one, and in this case the dual pairs are mutually isomorphic. They are irreducible iff } \mathcal{R}^- \text{ spans } (V^-)^*. \text{ The dual symmetric pairs are either both complex symmetric or both fail to be so. They are complex symmetric iff } \mathcal{R} \text{ spans a subspace of } V^* \text{ of codimension one and } \mathcal{R}^- \text{ spans } (V^-)^*. \text{ In this case the dual pairs are actually semikählerian.}\]

These facts are proven without classification.

**REFERENCES**

4. ———, *Classification of affine root systems and applications to the theory of symmetric spaces*, Mimeographed Notes, Washington University, St. Louis, Mo., 1968.

WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130 AND
ST. LOUIS UNIVERSITY, ST. LOUIS, MISSOURI 63103