RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

ON PROJECTIONS OF SELFADJOINT OPERATORS AND OPERATOR PRODUCT ADJOINTS

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Introduction. The proofs given in §§1 and 2 of this note were contained in a letter from the present writer to G. Lumer, sent shortly after the appearance of [1]. The subsequent appearance of [2] makes it desirable to publish these demonstrations. In [3] we mention an observation relevant to the topic of [2].

1. On [1]. [1] consists primarily of the proof of the following lemma and an addendum. Let $T$ be any selfadjoint operator in $\mathcal{H}$, a Hilbert space, $\mathcal{Y}$ a closed subspace in $\mathcal{H}$, $P$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{Y}$, $\chi = \mathcal{Y}^\perp$, the orthogonal complement of $\mathcal{Y}$ in $\mathcal{H}$. Let $T_0 = PT P$.

Lemma 1. If $\chi$ is finite dimensional, then $T_0$ is selfadjoint.

Our Proof. $\mathcal{D}(T_0) = \mathcal{D}(TP)$ is dense by [3, Theorem IV 2.7(iv), p. 103; note $\mathcal{D}(T)$ dense is the only property of $T$ used in (iv)]. Any time $\mathcal{D}(TP)$ is dense, $(T_0)^* = [P(TP)]^* = (TP)^* P \supseteq PTP = T_0$, and $T_0$ is selfadjoint iff $(TP)^* = PT$ iff $PT$ is closed, the latter implication seen from $(TP)^* = (T^*P^*)^* = (PT)^* \supseteq PT$, equality holding iff $PT$ is closed. But $PT$ is closed by [3, Theorem IV 2.7(i)].

The addendum of [1] asserts that a remark credited to G. Lumer in the acknowledgment of [4] is incorrect. However, the remark is correct in the context of bounded operators, which clearly was the context intended by Williams [4]. In that context, $T_0 = ATA$ is obviously selfadjoint whenever $A$ and $T$ are.

2. On [2]. Let $S$ and $T$ be densely defined closed linear operators in $\mathcal{H}$; [2] deals with the interesting question of when $(TS)^* = S^*T^*$

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is true beyond the well-known case where $T$ is bounded and everywhere defined, and adopting the ideas of [1] the following results are stated (proofs deferred to a detailed paper to be submitted elsewhere).

**Theorem.** If $T$ is a closed domain dense linear operator on $\mathcal{H}$, and $S$ is a bounded everywhere defined linear operator whose image is a closed subspace of finite codimension in $\mathcal{H}$, then $(TS)^* = S^*T^*$.

**Corollary.** If $S$ and $T$ are selfadjoint, and if $S$ is bounded, has a closed image, and has finite dimensional kernel, then $STS$ is selfadjoint.

Our proofs. The corollary is a special case of the theorem; alternately, replace $P$ by $S$ in the above proof. For the theorem $\mathcal{D}(TS)$ is dense as above, and $(TS)^* = (T^*S^*)^* = [(S^*T^*)^*] = (S^*T^*)^* \supseteq S^*T^*$, equality holding iff $S^*T^*$ is closed. But the latter is true by [3, Theorem IV 2.7(i)].

3. On when $(TS)^* = S^*T^*$. It should be observed, without further elaboration, that the question of [2] has a fairly general answer via the Fredholm theory (e.g., see [3]).

**Proposition 1.** If $T$ and $S$ are Fredholm operators, then $(TS)^* = S^*T^*$.

**Proof.** Let $\kappa$ denote the index of an operator. Since $(TS)^* \supseteq S^*T^*$, it is sufficient to demonstrate that $\kappa((TS)^*) = \kappa(S^*T^*)$. By the Fredholm theory, one has $\kappa((TS)^*) = -\kappa(TS) = -\kappa(T) - \kappa(S) = \kappa(T^*) + \kappa(S^*) = \kappa(S^*T^*)$.

**Additional Remark.** We have found that Proposition 1, with $T$ just closed, was shown in [5]; see also [6, Lemma 2.3]. Accordingly we state:

**Proposition 2.** Let $T$ and $S$ be densely defined linear operators in a Hilbert space. If $S$ is closed and $\mathcal{R}(S)$ has finite codimension, then $(TS)^* = S^*T^*$.

**Proof.** As in [5, Lemma 4.1]. More specifically: $T$ closed is not needed (e.g., by referring to [3]); $\mathcal{N}(S)$ and $\mathcal{R}(S)$ closed are implied by the hypotheses; dim $\mathcal{N}(S)$ need not be finite; and the bound [5, (4.3)] holds when $S$ and $\mathcal{R}(S)$ are closed.

Further ramifications are conceivable (e.g., assume a little more on $T$, a little less on $S$), as well as extensions to normed linear spaces, etc. On the other hand, Proposition 2 is rather sharp for arbitrary $T$ in that the conditions on $S$ are those generally required both to conclude that $(TS)^*$ exists and that $S^*T^*$ is closed.
REFERENCES


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