ON SOME SINGULAR CONVOLUTION OPERATORS

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Communicated by Joseph J. Kohn, February 14, 1969

In this note, we state some results on the boundedness of certain operators on $L^p(R^n)$. The operators which we study are too singular to be handled by the ordinary Calderón-Zygmund techniques of [1].

Our first theorem concerns a sublinear operator $g_\lambda^*$ which arises in Littlewood-Paley theory. If $f$ is a real-valued function on $R^n$, set $u(x, t)$ equal to the Poisson integral of $f$, defined on $R^{n+1}_+ = R^n \times (0, \infty)$. Then for $\lambda > 1$, the $g_\lambda^*$-function on $R^n$ is defined by the equation

$$g_\lambda(f)(x) = \left( \int_{R^{n+1}_+} \left( \frac{t}{|x-y|+t} \right)^{\lambda n} t^{1-n} |\nabla u(y, t)|^2 dy dt \right)^{1/2}.$$ 

($\nabla$ denotes the gradient in $R^{n+1}$.)

It is known [4] that if $p > 2/\lambda$ then the operator $f \mapsto g_\lambda^*(f)$ is bounded on $L^p(R^n)$. On the other hand, if $p < 2/\lambda$ then there are $L^p$ functions $f$ such that $g_\lambda^*(f)(x) = +\infty$ for every $x \in R^n$. The behavior of $g_\lambda^*$ on $L^p$ for $p = 2/\lambda$ is more subtle, and the methods of [1] and [4] are inadequate to deal with it.

**Theorem 1.** Let $1 < p < 2$, $\beta = 2/\lambda$. Then the operator $f \mapsto g_\lambda^*(f)$ has weak-type $(p, p)$, i.e.

$$\text{measure}(\{x \in R^n \mid g_\lambda^*(f)(x) > \alpha\}) \leq (A/\alpha^p)\|f\|_p^p$$

for any $\alpha > 0$ and $f \in L^p(R^n)$, and the "constant" $A$ is independent of $f$ and $\alpha$.

This result implies the positive theorem about $p > 2/\lambda$, for the case $p \leq 2$, by the Marcinkiewicz interpolation theorem.

An argument almost identical to the proof of Theorem 1 gives information on fractional integration. In particular, suppose that $f \in L^p(R^n)$ and $0 < \beta < 1$. Stein [5] has shown that the fractional integral $F = I^\beta(f)$ satisfies the smoothness condition

\[ This work was supported by the National Science Foundation. \]

\[ I am deeply grateful to my adviser and teacher, E. M. Stein, for bringing these problems to my attention and for his many helpful suggestions and criticisms. \]
\[ \mu_\beta(F) = \left( \int_{\mathbb{R}^n} \frac{|F(x) - F(x-y)|^2}{|y|^{n+2\beta}} \, dy \right)^{1/2} \in L^p(\mathbb{R}^n), \]

provided that \(2n/(n+2\beta) < p\); and that conversely, any function \(F \in L^p(\mathbb{R}^n)\) for which \(\mu_\beta(F)\) belongs to \(L^p\), has a fractional derivative \(I^{-\beta}F\) in \(L^p\). This result follows from the study of \(g_\lambda^\alpha\), since one can prove a pointwise inequality \(\mu_\beta(f)(x) \leq C_\lambda^\alpha(f)(x)\), for \(n(\lambda - 1) > 2\beta\), \(0 < \beta < 1\).

**Theorem 1'**. For \(1 < p < 2\), \(2n/(n+2\beta) = p\), and \(0 < \beta < 1\), the operator \(f \rightarrow \mu_\beta(I^p f)\) has weak-type \((p, p)\).

Theorem 1' is the best possible positive result for \(\mu_\beta\).

The above theorems exhibit various nonlinear operators which are bounded on some \(L^p\) spaces, but not on all. There are also some known examples of linear operators which are bounded only on some of the \(L^p\) spaces. For example, consider the operator

\[ T_{a\alpha} : f \rightarrow \left( \frac{\exp[i/|x|^\alpha]}{|x|^{n+a}} \right) * f, \]

defined for \(f \in C_0^\infty(\mathbb{R}^n)^*\). The convolution makes sense if we interpret \(\exp[i/|x|^\alpha]/|x|^{n+a}\) as a temperate distribution on \(\mathbb{R}^n\). Fix an \(\alpha > 0\) and an \(a > 0\). For which \(p\) does \(T_{a\alpha}\) extend to a bounded linear operator on \(L^p(\mathbb{R}^n)\)? If \(\alpha\) were negative, then \(k = \exp[i/|x|^\alpha]/|x|^{n+a}\) would be locally \(L^1\); so if we ignore difficulties at infinity (say by cutting off \(k\) outside of \(|x| < 1\)), we find that \(T_{a\alpha}\) is bounded on \(L^p\) for every \(p\) (\(1 \leq p \leq +\infty\)), if \(\alpha < 0\). On the other hand, by computing the Fourier transform of \(\exp[i/|x|^\alpha]/|x|^{n+a}\), we can deduce that \(T_{a\alpha}\) is bounded on \(L^2(\mathbb{R}^n)\) exactly when \(\alpha \leq (n/2)a\). (Since \(T_{a\alpha}\) is defined only on \(C_0^\infty(\mathbb{R}^n)\), the statement "\(T_{a\alpha}\) is bounded on \(L^p\)" means that \(T_{a\alpha}\) extends to a bounded operator on \(L^p\), or equivalently, that the a priori inequality \(\|T_{a\alpha}f\|_p \leq A\|f\|_p\) holds, for \(f \in C_0^\infty(\mathbb{R}^n)\).

Applying a strong form of the Riesz-Thorin convexity theorem, we can interpolate between the \(L^1\) inequality and the \(L^2\) inequality, to obtain the following theorem. Let \(\alpha > 0\), and let \(\beta = (a+1)(na/2-\alpha)\) be positive. (The significance of \(\beta\) is that it turns out that \(\left| \frac{\exp[i/|x|^\alpha]}{|x|^{\alpha}} \right|(y) = O(|y|^{-\beta})\) as \(|y| \to \infty\).) Then \(T_{a\alpha}\) is bounded on \(L^p(\mathbb{R}^n)\) if
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\[ \frac{1}{2} - \frac{1}{p} < \frac{\beta}{n} \left[ \frac{n/2 + \alpha}{\beta + \alpha} \right]. \]

Easy examples show that \( T_{a^*} \) cannot even have weak-type \((p, p)\) if

\[ \frac{1}{2} - \frac{1}{p} > \frac{\beta}{n} \left[ \frac{n/2 + \alpha}{\beta + \alpha} \right]. \]

The question has been raised, whether \( T_{a^*} \) is bounded on \( L^p(\mathbb{R}^n) \) where

\[ \left| \frac{1}{2} - \frac{1}{p_0} \right| = \frac{\beta}{n} \left[ \frac{n/2 + \alpha}{\beta + \alpha} \right]. \]

But no a priori \( L^p \) inequalities of any sort were known previously. We have proved the following partial result.

**Theorem 2.** Let \( \alpha, a \) and \( p_0 \) be as above, and let \( q_0 \) be the exponent conjugate to \( p_0 \). Then \( T_{a^*} \) extends to a bounded linear operator from \( L^p(\mathbb{R}^n) \) to the Lorentz space \( L^{p,q_0}(\mathbb{R}^n) \). (For an exposition of Lorentz spaces, see [3].)

Theorem 2 follows, using complex interpolation, from the two special cases \( p = 1 \) and \( p = 2 \). The case \( p = 2 \) is immediate from the Plancherel theorem, and the case \( p = 1 \) is just an example of the following generalization of the Calderón-Zygmund inequality.

**Theorem 2'.** Let \( K \) be a temperate distribution on \( \mathbb{R}^n \), with compact support; and let \( 0 < \theta < 1 \) be given. Suppose that \( K \) is a locally integrable function, away from zero, and that

(i) The temperate distribution \( \hat{K} \) is a function, and

\[ \left| \hat{K}(x) \right| \leq A(1 + |x|)^{-\theta} \quad \text{for } x \in \mathbb{R}^n. \]

(ii) \( \int_{|x| > |y|^{1-\theta}} |K(x) - K(x-y)| \, dx \leq A \) for all \( y \in \mathbb{R} \), \( |y| < 1 \).

Then the operator \( f \rightarrow K * f \), defined for \( f \in C_0^\infty(\mathbb{R}^n) \) extends to an operator \( T \) of weak-type \((1, 1)\).

Obviously, then, \( T \) is a bounded operator on \( L^p(\mathbb{R}^n) \), for \( 1 < p < \infty \).

A concrete example of a \( K \) satisfying (i) and (ii) is the kernel \( K(x) = \exp(i|x|^{1/\theta})/x \) for \( x \in \mathbb{R}^1 \), \( |x| < 1 \), and \( K(x) = 0 \) otherwise.

Theorem 2' can be strengthened in various ways. First of all, under reasonable assumptions on \( K \), we can prove a weak-type inequality for the "maximal operator"
\[ Mf(x) = \sup_{\varepsilon > 0} \left| \int_{|y| < \varepsilon} K(y)f(x-y)dy \right|. \]

Secondly, a proof almost identical to that of Theorem 2' establishes a weak-type inequality for convolutions with kernels whose singularities lie at infinity, instead of at zero.

For a discussion of \( T_{aa} \) and similar operators, see Hirschmann [2] for the one-dimensional case, and Wainger [7] and Stein [6] for the \( n \)-dimensional case.

The operators we have discussed so far are only slightly more singular than the Calderón-Zygmund operators of [1], or operators which reduce to them by interpolation. We now discuss \( L^p \) inequalities for highly singular operators, for which the techniques of [1], [4], and [6] break down completely.

Let \( T_{\alpha}: f \mapsto f * (\sin |x|/|x|^\alpha) \), for \( f \in C_0^\infty(\mathbb{R}^n) \). \( T_{\alpha} \) has an especially neat interpretation if \( \alpha = (n+1)/2 \). In fact, the operator \( S \), given by \( (Sf)(x) = \chi(x)f(x) \) (\( \chi \) denotes the characteristic function of the unit ball in \( \mathbb{R}^n \)), differs from \( T_{(n+1)/2} \) by an error term which is relatively small, so that, roughly speaking, \( S \) and \( T_{(n+1)/2} \) are the same.

It is easy to show that for \( p \leq 2n/(n+1) \) or \( p \geq 2n/(n-1) \), the operator \( S \) cannot be extended to a bounded operator on \( L^p(\mathbb{R}^n) \).

The question of whether \( S \) (or \( T_{(n+1)/2} \)) extends to a bounded operator on \( L^p(\mathbb{R}^n) \) for \( 2n/(n+1) < p < 2n/(n-1) \), or for that matter, for any \( p \) other than 2, is a well-known unsolved problem.

By interpolation between \( p = 2, \alpha = (n+1)/2 \), and \( p = 1, \alpha = n+\epsilon \), it is easy to prove that \( T_{\alpha} \) is bounded on \( L^p(\mathbb{R}^n) \), for

\[
\left( \frac{1}{p} - \frac{1}{2} \right) \left( \frac{n-1}{2} \right) < \alpha - \frac{n+1}{2}, \quad 1 < p < 2, \quad \frac{n+1}{2} < \alpha < n.
\]

See [6]. But we have every right to expect a far stronger inequality. For if we assume the conjecture that \( T_{(n+1)/2} \) is bounded on \( L^{2n/(n+1)+\epsilon}(\mathbb{R}^n) \), then it follows (at least heuristically) by interpolation, that \( T_{\alpha} \) is bounded on \( L^p(\mathbb{R}^n) \) for \( p \) in the larger range \( n/\alpha < p < 2 \), \( (n+1)/2 < \alpha < n \). This is the "right" range, since for \( p \leq n/\alpha \) it is easily seen that \( T_{\alpha} \) does not extend to a bounded operator on \( L^p(\mathbb{R}^n) \).

**Theorem 3.** Let \( n/\alpha < p < 2 \), and \( p < 4n/(3n+1) \). Then \( T_{\alpha} \) extends to a bounded linear operator on \( L^p(\mathbb{R}^n) \).

**References**


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