RULED SURFACES AND THE ALBANESE MAPPING

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Communicated March 17, 1969

1. Much of the classical theory of algebraic curves is summarized by saying there is a map \( C(n) \to J \) from the \( n \)-fold symmetric product of the curve \( C \) into an abelian variety \( J \), the Jacobian, and the fibers are projective spaces (representing the linear systems of degree \( n \)). For algebraic surfaces there is an analogous map \( V(n) \to A \) from the \( n \)-fold symmetric product of the surface \( V \) to its Albanese variety. The fibers are irreducible and regular if \( n \) is large, but it has been a long open question whether they are rational, or ever can be.

**Theorem.** Let \( V \) be a complete nonsingular surface in characteristic zero, and let \( q \) denote the dimension of its Albanese variety \( A \). If for some \( n > q \) the general fiber of the morphism \( V(n) \to A \) is a rational variety, then \( V \) is a ruled surface.

By the "general" fiber we mean as usual that there is an open set in \( A \) over which all fibers have the indicated property. If \( V \) is ruled, i.e., birationally equivalent to the product \( P^1 \times C \) of a projective line and a curve \( C \), then the general fiber is rational for all \( n \); for this converse to the theorem, one needs only the quoted result for curves plus the remark that then the Albanese variety of \( V \) is just the Jacobian of \( C \). A proof of the theorem when \( q = 0 \) was the subject of an earlier paper [2], some of whose ideas recur here. There is also overlap with a recent (independent) proof by Mumford [3] that the rational equivalence ring is not of finite type; both proofs use the idea of bounding the dimension of the zero-locus of a 2-form.

2. A generic smoothness lemma. We need the

**Lemma.** Let \( f: X \to Y \) be a dominating morphism of varieties in characteristic zero, with \( X \) nonsingular and projective. Then \( f \) has maximal rank along the general fiber \( F_y \), so \( F_y \) is nonsingular.

**Proof.** The lemma is local on \( Y \); by Noether normalization we may reduce to the case where \( Y \) is affine \( r \)-space, with coordinate functions \( x_1, \ldots, x_r \). As \( a_i \) varies over the (algebraically closed) ground field, the zeros of \( x_i - a_i \) on \( X \) give a linear system of divisors on \( X \); by Bertini's theorem, a general member—say \( X_1 \)—is a disjoint union of

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1 Research supported in part by the National Science Foundation.
nonsingular varieties; each has multiplicity one and no embedded components. In the same way the zeros of \( x_1 - a_0 \) on \( X_1 \) give a linear system on \( X_1 \) whose general member is nonsingular, etc. Continuing, we see that there exist fibers \( f^{-1}(a) \) along which \( f \) is of maximal rank. This then follows for the general fiber since \( f \) being of maximal rank is an open condition on \( Y \); it is clearly open on \( X \) and \( f \) is proper.

For the proof of our theorem, we may replace the singular variety \( V(n) \) by a canonical desingularization, the Hilbert scheme \( H(n) \). This is [1] a nonsingular 2\( n \)-dimensional projective variety with a birational morphism \( h_n : H(n) \to V(n) \). This gives a corresponding morphism \( H(n) \to A \), whose general fiber is pure \((2n - q)\)-dimensional by dimension theory, and it is easily seen that \( h_n \) is an isomorphism on an open subset of each component of the general fiber. Thus our hypotheses together with the lemma imply that

(1) For some \( n > q \), the general fiber of \( H(n) \to A \) is a nonsingular rational variety.

3. Differential forms of weight \( r \). Let \( X \) be an \( n \)-dimensional variety, with function field \( K = k(X) \), and let \( E^p \) be the \( K \)-space of \( p \)-forms on \( X \); then the \((m_1, \ldots, m_r)\)-forms are the elements of \( E^{m_1} \otimes \cdots \otimes E^{m_r} \); if all \( m_i = m \), they are called the \( m \)-forms of weight \( r \). These forms are holomorphic at \( p \in X \) if the coefficients are holomorphic when the form is written in terms of \( dx_1, \ldots, dx_n \), where the \( x_i \) are local parameters at \( p \). Thus if \( X \) is also complete, the number of independent global holomorphic \((m_1, \ldots, m_r)\)-forms is given by \( h_{m_1, \ldots, m_r} = \dim \mathcal{H}^0(X, \Omega^{m_1} \otimes \cdots \otimes \Omega^{m_r}) \), where \( \Omega \) is the sheaf of holomorphic \( k \)-forms. If \( m_i = n \) for all \( i \), \( h_{n^n} \) is traditionally written \( P_r(X) \), and called the \( r \)th plurigenus of \( X \). These are all birational invariants for \( X \) complete nonsingular, for

(2) If \( f : X \to Y \) is a dominating, separable, rational map of complete nonsingular varieties, then \( h_{m_1, \ldots, m_r}(X) \geq h_{m_1, \ldots, m_r}(Y) \).

The reasoning is classical. If \( \alpha \) is a holomorphic form on \( Y \), then \( f^*\alpha \) is a form of the same type on \( X \) which is nonzero (separability); holomorphic outside a locus of codimension \( \geq 2 \) (the fundamental locus), therefore holomorphic everywhere (nonsingularity of \( X \)).

PROPOSITION. If \( X \) is a nonsingular rational (or unirational) variety, then \( h_{m_1, \ldots, m_r}(X) = 0 \) for all \( (m_1, \ldots, m_r) \neq (0, \ldots, 0) \).

PROOF. It suffices to prove this when \( X \) is projective \( n \)-space, by (2). Let \( x_0, \ldots, x_n \) be projective coordinates, and \( g : A - (0) \to X \) the usual map of affine \((n+1)\)-space minus the origin onto projective
space. If \( \alpha \) is a holomorphic form on \( X \), then \( g^*\alpha \) is holomorphic on \( A - (0) \), therefore on \( A \) since \( \text{cod } (0) \geq 2 \). Written in terms of \( x_i \), its coefficients are thus polynomials; since it is invariant under the automorphisms of \( A \) defined by \( x_i \rightarrow cx_i \), we get all \( m_i = 0 \).

4. Proof of the theorem. By a well-known result (see e.g. [4]), if \( V \) is a complete nonsingular surface in characteristic zero, then \( V \) is ruled if and only if \( P_r(V) = 0 \) for all \( r > 0 \). So we prove:

(3) If for some \( r \), \( V \) carries a nonzero holomorphic 2-form \( \phi \) of weight \( r \), then the general fiber of \( H(n) \) is not rational.

Let \( V[n] \) be the \( n \)-fold product; given such a \( \phi \), then

(4) \( \Phi = \phi_1 + \cdots + \phi_n, \quad \phi_i = \text{pr}_i^*\phi \)

is a holomorphic 2-form of weight \( r \) on \( V[n] \); since it is invariant under the symmetric group \( S_n \), it is the lifting of a form on \( V(n) \), and this in turn may be carried over to \( H(n) \). We use the same letter \( \Phi \) for any of these forms. If we grant that \( \Phi \) is holomorphic on \( H(n) \)—this will be proved later—then the restriction \( \Phi_F \) of \( \Phi \) to a (nonsingular) general fiber \( F \) of \( H(n) \) gives a holomorphic 2-form of weight \( r \) on \( F \). If \( F \) were rational, then \( \Phi_F = 0 \) by the proposition. But if we pull things back to \( V[n] \), this contradicts

(5) If \( n > q \), the restriction of \( \Phi \) to the general fiber of \( V[n] \) is not zero.

Proof of (5). Let \( p = (p_1, \ldots, p_n) \) be a general point of \( V[n] \), \( F \) the fiber through it, \( T_{p,F} \) the tangent space to \( F \) at \( p \). We say

(6) \( \sigma_i : T_{p,F} \rightarrow T_{p_i,F} \) is onto for all \( i \) (\( \sigma_i = d(\text{pr}_i|F) \)).

Namely, let \( S_i \) be the closure of the set of the fiber \( F_q \) through them or else where \( \sigma_i \) is not onto, i.e., has rank \( \leq 1 \). Since \( \dim T_{p,F} = 2n - q > n \), this space cannot be mapped to a 1-dimensional space by each of the \( n \) maps \( \sigma_i \). Say \( \sigma_i \) has rank 2; introducing coordinates, we see that rank \( \sigma_i = 2 \) in a neighborhood of \( p \). Thus \( p \in S_i \), so \( S_i \) is a proper closed set. It follows by symmetry that \( S_i \) is a proper closed set, and therefore \( p \in S_i \) for any \( i \), which is the assertion (6).

From (6) it follows that for each \( i \), we can choose vectors \( t_i, t'_i \) in \( T_{p,F} \) whose images under \( \sigma_i \) are independent. Taking general linear combinations of the \( t_i \) and of the \( t'_i \), we conclude

(7) There are vectors \( t, t' \) in \( T_{p,F} \) such that \( \sigma_i(t) \) and \( \sigma_i(t') \) are independent for all \( i \).
We now prove (5). Choose $x$ and $y$ to be local parameters at each point $p_i$; thus $\phi = g(dx dy)\big|_r^r$, where $g(p_i) = a_i \neq 0$ since $p_i$ is a general point of $V$. By (7), $(dx dy, (\sigma_i(t), \sigma_i(t'))) = b_i \neq 0$. On the space $T_{p,V[n]}$, by (4) the form $\Phi = \sum a_i(dx_i dy_i)\big|_r^r$. If $\Phi$ were 0 when restricted to the subspace $T_{p,F}$, then for $e, e' \in T_{p,F}$,

$$\langle \Phi, (e, e', t, t', \cdots, t, t') \rangle = \sum a_i(dx_i dy_i, (e, e')) b_i^{-1} = 0.$$  

Our hypothesis is that $\dim T_{p,F} > n$. If we put in $n+1$ linearly independent vectors for $e'$, we get from the above $n+1$ independent linear equations in $2n$ variables (the coefficients of $e$), having at least $n+1$ independent solution vectors $e$, a contradiction.

We still must show $\Phi$ is holomorphic on $H(n)$. Let $X$ be the normalization of $H(n)$ in the function field of $V[n]$. Then the symmetric group $S_n$ acts as automorphisms of $X$ and $H(n)$ is the quotient $X/S_n$. Since $\Phi$ is holomorphic on $V[n]$, when viewed as a differential $\Phi'$ on the normal and birationally equivalent variety $X$, it will have no poles. Therefore on $H(n)$, its trace $\text{tr}_{X/H(n)} \Phi'$ will also have no poles; but $(1/n!) \text{tr} \Phi = \Phi$.

REFERENCES