THE SPECTRUM OF NONCOMPACT $G/\Gamma$ AND THE COHOMOLOGY OF ARITHMETIC GROUPS

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Communicated by Louis Auslander, February 13, 1969

Introduction. The purpose of this note is to announce a theorem in the representation theory of semisimple groups (Theorem 1.2, below). This theorem implies that certain spaces of square summable harmonic forms on noncompact locally symmetric spaces, associated with $\mathcal{Q}$-rank one arithmetic groups, are finite dimensional. Assertion (1.3) then gives information about the boundary behavior at $\infty$ of such forms. Combining (1.3) with the computations in [4] and Raghunathan's square summability criterion in [6], we obtain upper bounds for some betti numbers of locally symmetric spaces associated with $\mathcal{Q}$-rank one arithmetic groups (these spaces are noncompact, but have the homotopy type of a finite simplicial complex (see [7])). In some cases we obtain vanishing theorems for the first and second betti numbers. For the first betti number, such a vanishing theorem was obtained in greater generality by D. A. Kazdan (see [3]) by a different method. We remark that Raghunathan's square summability criterion has been generalized to arbitrary $\mathcal{Q}$-rank in [1]. Therefore an extension of Theorem 1.2 to arbitrary $\mathcal{Q}$-rank would yield a corresponding extension of our present results on cohomology. A detailed proof of Theorem 1.2 and a full discussion of the application of this theorem to the cohomology of arithmetic groups will appear elsewhere. I wish to express my thanks to S. T. Kuroda and M. S. Raghunathan for stimulating discussions.

We now introduce some notation. Let $\mathcal{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers, respectively, and let $\mathbb{Z}$ denote the ring of rational integers. Let $G$ denote a connected, linear, semisimple, algebraic group which is defined and simple over $\mathcal{Q}$. For a subring $A \subseteq \mathbb{C}$, let $G_A$ denote the $A$-rational points of $G$. However, when $A = \mathbb{R}$, we let $G = G_\mathbb{R}$. We let $\mathfrak{g}$ denote the Lie algebra of $G$, $\mathfrak{g}_\mathbb{C}$ the complexification of $\mathfrak{g}$, and $\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. We make the convention that $\mathfrak{g}$ is the space of right invariant vector fields on $G$. Hence $\mathfrak{g}$ is the space of right invariant differential operators on $G$. We denote the center of $\mathfrak{g}$ by $\mathfrak{z}$. As is well known, $\mathfrak{z}$ may be identified with the space of (adjoint-)invariant polynomials.

1 The author was partially supported by NSF Grant GP-7131 and a Yale University Junior Faculty Fellowship.
on $\mathfrak{g}_C$. In particular, there is a unique element $\Delta_\theta \in \mathcal{B}$, called the Casimir operator, which corresponds to the Killing form under this identification.

Let $\Gamma \subset G$ be an arithmetic subgroup. We fix a Haar measure $\text{d}v$ on $G$, and note that $\text{d}v$ induces a $G$-invariant measure on $G/\Gamma$ (which we again denote by $\text{d}v$). We let $L_2 = L_2(G/\Gamma)$ denote the space of $\mathbb{C}^\infty$, $\mathbb{C}$-valued functions $f$ on $G/\Gamma$, such that

$$\int_{G/\Gamma} f(x)f^-(x)\text{d}v(x) < \infty$$

(where $-\cdots$ denotes complex conjugation).

We fix a maximal $\mathfrak{Q}$-split torus $\mathfrak{S}_G \subset \mathfrak{g}_G$, and let $\mathfrak{A}$ denote the topological identity component of the $\mathfrak{R}$-rational points of $\mathfrak{S}_G$. We let $Z(\mathfrak{Q}_G)$ denote the centralizer of $\mathfrak{Q}_G$ in $G$, and we let $X(\mathfrak{Q}_G)$ denote the $\mathfrak{Q}$-rational characters of $Z(\mathfrak{Q}_G)$. We then define $M \subset Z(\mathfrak{Q}_G)$ by

$$M = \bigcap_{\chi \in \chi(\mathfrak{Q}_G)} \text{kernel } \chi^2.$$ 

$Z(\mathfrak{Q}_G)$ is known to have an almost direct product decomposition $Z(\mathfrak{Q}_G) = M \mathfrak{Q}_G$, and $Z(\mathfrak{Q}_A)$, the centralizer of $\mathfrak{Q}_A$ in $G$, a direct product decomposition

$$Z(\mathfrak{Q}_A) = M \mathfrak{Q}_A,$$

where $M$ denotes the $\mathfrak{R}$-rational points of $M$.

We now fix a maximal compact subgroup $K \subset G$, such that $K$ and $\mathfrak{Q}_A$ have Lie algebras which are orthogonal with respect to the Cartan-Killing form of $\mathfrak{g}$. Let $V$ be a finite dimensional, complex vector space with a positive definite, Hermitian inner product. Then let $\sigma : K \rightarrow \text{Aut } V$ be a representation of $K$ which is unitary with respect to the given inner product. We let $d_\sigma$ denote the complex dimension of $V$ and we let $\xi_\sigma$ denote the character of $\sigma$.

We then define a subspace $L'_2$ of $L_2$, by

$$(0.1) \quad L'_2 = \left\{ f \in L_2 \mid d_\sigma \int_{\mathcal{K}} \xi_\sigma(k)f(k^{-1}x)dk = f(x), \quad x \in G/\Gamma \right\},$$

where $dk$ denotes Haar measure on $\mathcal{K}$, normalized so that

$$\int_{\mathcal{K}} dk = 1.$$ 

We remark that functions on $G/\Gamma$ may be identified with $\Gamma$-invariant functions on $G$. We will make this identification whenever convenient.
and we will denote corresponding functions on $G$ and $G/T$ by the same letter.

1. **Statement of the main theorem.** For $v \in \mathcal{C}$, let

$$\mathcal{G}^* = \{ f \in L_2 | \Delta f = vf \}.$$

**Lemma 1.1.** Assume $G$ has $Q$-rank one; i.e. $\dim Q = 1$. Then there exists a real number $J$ so that if $\mathcal{G}^* \neq \{ 0 \}$, then $v$ is real and $v < J$.

**Theorem 1.2 (Main theorem).** Assume $G$ has $Q$-rank one. For $c \in \mathbb{R}$, let

$$\mathfrak{F}^* = \oplus_{c > J} \mathcal{G}^*.$$ Then $\mathfrak{F}^*_c$ is finite dimensional. Moreover, if $v \in \mathbb{R}$, $f \in \mathcal{G}^*$ and $\Lambda \in \mathfrak{G}$, we have $\Delta f \in L_2$. If $\nu_1, \nu_2 \in \mathbb{R}$, $f_1 \in \mathcal{G}^*$, and $\Lambda_1, \Lambda_2 \in \mathfrak{G}$, then for $X \in \mathfrak{g}$, where we have

$$\int_{\mathfrak{g}/\mathfrak{t}} (X\Delta f_1)(\Delta_2 f_2) dv = -\int_{\mathfrak{g}/\mathfrak{t}} (\Delta_1 f_1)(X\Delta_2 f_2) dv.$$

The following is an immediate consequence of Lemma 1.1 and Theorem 1.2.

**Corollary 1.4.** The eigenvalues of $\Delta_Q$ in $L_2^*$ have no finite point of accumulation.³

2. **An indication of the proof of the main theorem.** In this section we assume $G$ has $Q$-rank one. Let $P \subset G$ be a minimal $Q$-parabolic subgroup and let $P$ denote the $R$-rational points of $P$. We let $U$ denote the unipotent radical of $P$ and $U$ the $R$-rational points of $U$. After conjugating $P$ by a suitable point in $G$, we can assume

$$P = M_Q SU, \quad P = M_Q A U.$$ We let $\mathcal{Z}$ denote a set of double coset representatives for $P_Q \backslash G_Q / \Gamma$, and we let

$$\Gamma_\infty = \bigcap_{q \in \mathcal{Z}} q \Gamma q^{-1} \cap U.$$ $U/\Gamma_\infty$ is compact, and we can therefore fix a Haar measure $du$ on $U$ so that $\int_{U/\Gamma_\infty} du = 1$. For $f \in L_2$ and $g \in \mathcal{Z}$, we define $f_g$ by $f_g(x) = f(xg)$, $x \in G$ (if here being identified with a right $\Gamma$ invariant func-

³ At first we proved $\mathcal{G}^*$ finite dimensional. We thank R. P. Langlands for pointing out that our argument also gives the finite dimensionality of $\mathfrak{F}^*_c$, and hence Corollary 1.4.
tion on $G$). We then define $f_q'$ by

$$f_q'(x) = \int_{U/F} f_q(xu)du, \quad x \in G.$$  

From now on, we assume $f \in C_G^\infty$ for some $\nu \in \mathbb{R}$ and some $\sigma$. In particular, $f \in L^2_\nu$ and this means that $f$ is a component of a $V$-valued, left $K$ equivariant function. The same is then true of $f_q'$. Moreover, since $G$ has the generalized Iwasawa decomposition

$$G = KM_QA U,$$

and since $f_q'$ is also right $U$ invariant, we see that $f_q'$ is uniquely determined by its restriction to $M_QA$. We denote this restriction again by $f_q'$.  

Recall that $M_QA$ is a direct product. We can therefore regard $f_q'$ as a function of two variables (the $M$-variable and the $Q_A$-variable). A central step in proving Lemma 1.1 and Theorem 1.2 is to determine the nature of $f_q'$ as a function of the $Q_A$-variable. For we can then apply the theory of cusp forms (see [2, Chapter 1]) together with arguments from the theory of elliptic operators (see [5]) to obtain the desired results. We will describe $f_q'$ as a function in the $Q_A$-variable presently, but in preparation, we introduce some notation. 

We let $\pi: MU \to M$ denote the natural projection. We let

$$\Gamma_P = \bigcap_{q \in \mathbb{Z}} (q\Gamma q^{-1} \cap MU), \quad \Gamma_M = \pi(\Gamma_P).$$

For each $a \in Q_A$, we set $f_{q,a}^\prime(m) = f_{q}^\prime(ma)$, $m \in M$. $f_{q,a}^\prime$ is then a right $\Gamma_M$-invariant function on $M$. Moreover, $\Gamma_M$ is a discrete subgroup of $M$ and $M/\Gamma_M$ is compact. Hence $f_{q,a}^\prime$ may be regarded as a function on the compact quotient space $M/\Gamma_M$. We let $K_M = \pi(K \cap MU)$ and we define $\sigma_M: K_M \to \text{Aut } V$, by

$$\sigma_M(\pi(k)) = \sigma(k), \quad k \in K \cap MU.$$ 

We then fix a Haar measure $dm$ on $M$, and define $L_2(M/\Gamma_M)$ and $L^2(H_M(M/\Gamma_M))$ just as we did $L_2(G/\Gamma)$ and $L^2(H_G(G/\Gamma))$, respectively. We note that $f_{q,a}^\prime \in L_2^M(M/\Gamma_M)$, for all $a \in Q_A$. The pair $(Q_A, U)$ determines an order on the roots of $Q_A$. We then let $\alpha$ denote the unique simple root and $Qg$ one half the sum of the positive roots. The behaviour of $f_q'$ as a function in $a$, $a \in Q_A$, is then given by

**Lemma 2.1.** There is an orthonormal basis $\phi_1, \ldots, \phi_i, \ldots$ of $L^2_M(M/\Gamma_M)$, a sequence of real numbers $m_1, \ldots, m_i, \ldots$ such that
Limit $t \to \infty$ $m_i = \infty$, and a positive number $\lambda$ depending only on $q$, so that if $v \in \mathbb{C}$ and $G_v \neq \{0\}$, then $v \in \mathbb{R}$ and there is a finite subsequence $\phi_{i_1}, \ldots, \phi_{i_N}$ with $m_{i_j} + \nu > 0$, $j = 1, \ldots, N$, so that if $\kappa = \lambda^{-1}(m_{i_j} + \nu)^{1/2}$ (here we take the positive square root), then for all $f \in \mathfrak{B}$, $q \in \mathbb{Z}$, we can find $b_1, \ldots, b_n \in \mathbb{C}$, so that

$$\exp(\tilde{g}(\log a))\phi_i(m^*a) = \sum_{j=1}^N b_j \exp(\kappa \alpha(\log a))\phi_i(m), \quad a \in \mathfrak{g}A, \ m \in M.$$ 

Here $\log a$ is the unique element in the Lie algebra of $\mathfrak{g}A$ which exponentiates to $a$.

**Remark.** The $\phi_i$ and $m_i$ are respectively the eigenfunctions and corresponding eigenvalues of a certain (essentially) elliptic invariant differential operator on $L^2(M/T)$ associated with $\Delta_\mathfrak{g}$.

**Bibliography**


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