ON SIMULTANEOUS APPROXIMATION AND INTERPOLATION WHICH PRESERVES THE NORM

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In [6] H. Yamabe established the following “simultaneous approximation and interpolation” theorem, which generalized a result of Walsh [4, p. 310] (cf. also [1], [3] for further generalizations), and is related to a theorem of Helly in the theory of moments (cf. e.g. [2, pp. 86–87]).

THEOREM (YAMABE). Let \( M \) be a dense convex subset of the real normed linear space \( X \), and let \( x_1^*, \ldots, x_n^* \in X^* \). Then for each \( x \in X \) and each \( \epsilon > 0 \), there exists a \( y \in M \) such that \( \| x - y \| < \epsilon \) and \( x_i^*(y) = x_i^*(x) \) (\( i = 1, \ldots, n \)).

Wolibner [5], in essence, proved that Yamabe’s theorem could be sharpened in the particular case when \( X = C([a, b]) \), \( M = \sigma = \) “the polynomials,” and the \( x_i^* \) are “point evaluations.” Indeed, from the results of [5] there can readily be deduced the following

THEOREM (WOLIBNER). Let \( a \leq t_1 < \ldots < t_n \leq b \) and let \( \sigma \) be the set of polynomials. Then for each \( x \in C([a, b]) \) and each \( \epsilon > 0 \), there exists a \( p \in \sigma \) such that \( \| x - p \| < \epsilon \), \( p(t_i) = x(t_i) \) (\( i = 1, \ldots, n \)), and \( \| p \| = \| x \| \).

Motivated by Wolibner’s theorem, we consider the following more general problem. Let \( M \) be a dense subspace of the real normed linear space \( X \), and let \( \{ x_1^*, \ldots, x_n^* \} \) be a finite subset of the dual space \( X^* \). The triple \( (X, M, \{ x_1^*, \ldots, x_n^* \}) \) will be said to have property SAIN (simultaneous approximation and interpolation which is norm-preserving) provided that the following condition is satisfied:

For each \( x \in X \) and each \( \epsilon > 0 \) there exists a \( y \in M \) such that \( \| x - y \| < \epsilon \), \( x_i^*(y) = x_i^*(x) \) (\( i = 1, \ldots, n \)), and \( \| y \| = \| x \| \).

In this note we shall outline some of the main results we have obtained regarding property SAIN. Detailed proofs and related matter will appear elsewhere.

THEOREM 1. Let \( M \) be a dense subspace of the Hilbert space \( X \) and let \( x_1^*, \ldots, x_n^* \in X^* \). Then \( (X, M, \{ x_1^*, \ldots, x_n^* \}) \) has property SAIN if and only if each \( x_i^* \) attains its norm on the unit ball in \( M \).

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The necessity in Theorem 1 is valid in any reflexive Banach space $X$. Whether the sufficiency is also valid in any reflexive Banach space is an open question. Also, in the case when $n=1$, Theorem 1 is valid in any strictly convex reflexive Banach space.

Let $T$ denote a compact Hausdorff space and $C(T)$ the real continuous functions on $T$ with the sup norm. If $t \in T$, $\delta_t$ will denote the functional "point evaluation" at $t$, i.e. $\delta_t(x) = x(t)$ for all $x \in C(T)$.

**Theorem 2.** Let $A$ be a dense subalgebra of $C(T)$ and $t_1, \ldots, t_n \in T$. Then $(C(T), A, \{\delta_{t_1}, \ldots, \delta_{t_n}\})$ has property SAIN.

Theorem 2 contains that of Wolibner and represents a strengthening of the Stone-Weierstrass theorem. Theorem 2 is proved by a rather tedious induction on $n$ using Yamabe's theorem and the following lemma which essentially allows us to approximate the unit function in a useful manner.

**Lemma.** Let $A$ and $t_i$ be as in Theorem 2. Then for each $\epsilon > 0$, there exists an element $e \in A$ such that $\|e - 1\| < \epsilon$, $e(t_i) = 1$ ($i = 1, \ldots, n$), and $e \leq 1$.

Theorem 2 is also valid if "dense subalgebra" is replaced by "dense linear sublattice containing constants." However, the following examples show that these results cannot be extended very far.

**Example 1.** Let $M = \{x \in C([0, 1]): x'(0)$ exists, $x'(\frac{1}{2}) = x(0) - x(1)\}$. Then $M$ is a dense subspace of $C([0, 1])$, which contains constants, but such that $(C([0, 1]), M, \delta_{T_2})$ does not have property SAIN (since if $x \in C([0, 1])$ is the function which is 1 if $0 \leq t \leq \frac{1}{2}$ and $x(t) = -2t + 2$ if $\frac{1}{2} < t \leq 1$, and $y$ is any element of $M$ which satisfies $y'(\frac{1}{2}) = 1$ and $\|y\| = \|x\| = 1$, then $y'(0) = 0$ so $y(0) = y(1)$ and hence $\|x-y\| = \|x\| = 1$).

**Example 2.** Let $A = \operatorname{span}\{x_1, x_2, \ldots\}$ where $x_i(t) = t^i$ ($i = 1, 2, \ldots$) and define $x^*$ by $x^*(x) = \int_0^1 x(t)dt$ for all $x \in C([1, 2])$. Then $A$ is a dense subalgebra of $C([1, 2])$ but $(C([1, 2]), A, x^*)$ does not have property SAIN (since if $e$ is the unit function, then any $y \in A$ which satisfies $x^*(y) = x^*(e) = 1$ must necessarily satisfy $\|y\| > 1 = \|e\|$).

**Example 3.** Let $L = \{x \in C([0, 1]): x'(0)$ exists, $x'(0) = x(0)\}$. Then $L$ is a dense linear sublattice in $C([0, 1])$ but $(C([0, 1]), L, \delta_0)$ does not have property SAIN (since if $e$ is the unit function and $y$ is any element of $L$ satisfying $y(0) = e(0) = 1$, then $y'(0) = y(0) = 1$ and
so \( y(t) > 1 \) for some \( t > 0 \) and hence \( \| y \| = \| e \| ).

In the case when \( X = L_p = L_p(T, \Sigma, \mu) \) (\( 1 < p < \infty \)) and \( M \) is the subspace of \( L_p \) consisting of those functions which vanish off sets of finite measure, we can prove the following theorem. (Recall that the représenter of a functional \( x^* \in L_p^* \) is the function \( y \in L_q, q = p/(p-1) \), such that \( x^*(x) = \int_T xy \, d\mu \) for all \( x \in L_p \).)

**Theorem 3.** Let \( 1 < p < \infty \), let \( M \subseteq L_p \) be as above, and let \( x_1^*, \ldots, x_n^* \). Then the following statements are equivalent.

1. \( (L_p, M, \{x_1^*, \ldots, x_n^* \}) \) has property SAIN.
2. Each \( x_i^* \) attains its norm on the unit ball in \( M \).
3. The représenter of each \( x_i^* \) vanishes off a set of finite measure.

**References**


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