A “FUNCTIONAL EQUATION” FOR MEASURES AND
A GENERALIZATION OF GAUSSIAN MEASURES

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1. Introduction. Let $G$ be a LCA group for which the map $x \mapsto 2x$ is an automorphism, and let $\xi : G \times G \to G \times G$ be defined by $\xi(x, y) = (x + y, x - y)$. We call a regular complex-valued measure $\mu$ on $G$ Gaussian iff there is a second measure $\nu$ on $G$ such that for all Borel sets $E \subseteq G \times G$,

\[(\mu \times \nu)(E) = (\nu \times \nu)(\xi(E)).\]

One rationale for this definition is that any finite probability measure on $\mathbb{R}$ which satisfies (1.1) is a Gaussian distribution with mean 0. (See [1, p. 77] for a proof.) Another reason is that the 2-adic theta functions defined by Mumford in [2] are related to 2-adic measures satisfying (1.1) much as ordinary theta functions are related to the Gaussian distribution $\exp(-ax^2)dx$.

Actually, we shall consider all set functions which are finite complex linear combinations of regular measures on $G$. These need not be $\sigma$-additive measures (since the regular measures need not be bounded), but we shall use the term measure for such functions as well.

The problem we consider is that of determining all Gaussian measures on $G$. In [2], Mumford did this in the case $G = (\mathbb{Q}_2)^n$; in §§3 and 4 of this paper we state the results for $G = \mathbb{R}^n$ and for $G$ a compact group. One reason for considering these cases is given by the following structure theorem.

**Theorem 1.** If $G$ is a LCA group such that $x \mapsto 2x$ is an automorphism, then $G$ can be written as $V \times W \times G_0$, where $V$ is a real vector group, $W$ is a 2-adic vector group, and $G_0$ contains a compact open subgroup for which $x \mapsto 2x$ is an automorphism.

Another attack on the problem is considered in §2, where we consider Gaussian measures which are absolutely continuous (with respect to Haar measure). The rationale behind this approach is the following result.

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1 The results announced here are contained in the author’s Ph.D. thesis at Harvard University, written while he held an N.S.F. Graduate Fellowship.
THEOREM 2. If \( \mu \) is Gaussian, then \( \mu \) is either absolutely continuous or singular.

However, the singular Gaussian measures are in general hard to analyze.

2. The absolutely continuous case. Let \( \mu, \nu \), be absolutely continuous measures on \( G \) satisfying (1.1), and let \( f, g \) be their respective Radon-Nikodym derivatives. Then \( f \) and \( g \) satisfy

\[
f(x)f(y) = g(x+y)g(x-y) \quad \text{a.e. on } G \times G.
\]

THEOREM 3. Let \( f, g \) be nonnull functions satisfying (2.1). Then there is an open subgroup \( G_0 \) of \( G \), closed under division by 2, such that \( f \) and \( g \) are nonzero a.e. on \( G_0 \) and are zero a.e. off \( G_0 \). Moreover, there are continuous functions \( f_0, g_0 : G_0 \to \mathbb{C}^\times \), and a complex number \( c_0 \), such that:

(a) \( c_0 f_0 = f, \quad \pm c_0 g_0 = g \) a.e. on \( G \), for some fixed choice of sign.

(b) \( f_0 \) and \( g_0 \) are quadratic characters (i.e., \( f_0(x+y)/f_0(x)f_0(y) \) is bilinear in \( x \) and \( y \), and similarly for \( g_0 \)).

It follows from continuity that \( f_0 \) and \( g_0 \) satisfy (2.1) everywhere on \( G_0 \) and hence that \( g_0(x)^2 = f_0(x) \).

To show that \( G_0 \) exists is fairly straightforward; in what follows, we assume that \( G_0 = G \). If (2.1) held everywhere, and \( f, g \) were never 0, we would proceed as follows: let \( c_0 = f(0) \), and let \( f_0 = f/c_0, \quad g_0 = g/\pm c_0 \), where the sign is chosen so that \( g_0(0) = 1 \). Then \( f_0 \) and \( g_0 \) satisfy (2.1). Let \( y = 0 \); we get \( g_0(x)^2 = f_0(x) \); now let \( x = 0 \) to show that \( g_0(y) = g_0(-y) \). Hence \( g_0 \) and \( f_0 \) are even functions. Substituting in (2.1), we find that

\[
f_0(x + y)f_0(x - y) = f_0(x)f_0(y)^2.
\]

This is essentially the parallelogram law; it follows without much trouble that \( f_0(x+y)/f_0(x)f_0(y) \) is bilinear.

In the actual theorem, substituting specific values for \( x \) and \( y \) in (2.1) is invalid. Instead, we use limit arguments, based on the density theorem and Lusin's theorem, to get the result.

3. Gaussian measures on compact groups. We begin by reducing the problem to a special case. Let \( f \) be the Fourier-Stieltjes transform of the Gaussian measure \( \mu \); an easy argument shows that \( f \) has support on a subgroup \( \Gamma \subseteq G^\times \) closed under division by 2 and that \( f \) is a multiple of a quadratic character on \( \Gamma_0 \). We may assume that \( f(0) = 1 \). Set \( G_0 = \Gamma_0^\times \). Then there is a Gaussian measure \( \mu_0 \) on \( G_0 \) (which we
shall call the condensation of \( \mu \) whose Fourier-Stieltjes transform is \( f \uparrow \Gamma_0 \). It is easy to obtain either \( \mu \) or \( \mu_0 \) in terms of the other; thus for convenience, we shall assume that \( \Gamma_0 = G^\wedge \).

Suppose that \( G^\wedge \) is a torsion-free group of rank \( n \). (Since \( x \rightarrow 2x \) is an automorphism of \( G \), \( G^\wedge \) is a \( \mathbb{Z}[\frac{1}{2}] \)-module.) Then we can define a Gaussian measure on \( G \) as follows: \( G \) contains a dense image of \( \mathbb{R}^n \), and any finite Gaussian measure on \( \mathbb{R}^n \) therefore defines a Gaussian measure on \( G \). Let \( B \) be any symmetric complex matrix whose real part is positive definite, and let

\[
\frac{1}{(\text{Det } B)^{1/2}} \exp((\pi B^{-1}x, x))dx.
\]

(Here, \(( , )\) is the usual scalar product, and the sign of \( B \) must be chosen so that \( \mu_B(\mathbb{R}^n) = 1 \).) We call such a measure on \( G \) a matrix measure. More generally, we call a Gaussian measure \( \mu \) on \( G \) matricial iff

(a) \( \mu^\wedge \) is never 0;

(b) \( \mu \) is the weak-* limit of Gaussian measures \( \mu \) whose Fourier-Stieltjes transforms are concentrated on subgroups \( H \) of finite rank and whose condensations are matrix measures on \( H^\wedge \).

The opposite extreme from a matricial measure is a large measure. We say that the Gaussian measure \( \mu \) is large iff \( |f(x)| = 1 \) for all \( x \in G^\wedge \). A more useful characterization is the following

**Theorem 4.** \( \mu \) is a large Gaussian measure \( \iff \mu \) is a Gaussian point measure concentrated on a finite subgroup of \( G \).

We can now state the characterization of Gaussian measures on \( G \).

**Theorem 5.** Let \( \mu \) be a Gaussian measure on \( G \) whose Fourier transform never vanishes. Then there are measures \( \mu_1 \) and \( \nu \), and a subgroup \( G_1 \), such that:

1. \( \mu_1 \) is concentrated on \( G_1 \); as a measure on \( G_1 \), \( \mu_1 \) is matricial;
2. \( \nu \) is a large measure;
3. \( \mu_1 \ast \nu = \mu \).

Moreover, (1), (2), and (3) uniquely determine \( \mu_1 \), \( \nu \), and \( G_1 \).

The proofs of these theorems involve getting restrictions on \( f \) in terms of \( \|\mu\| \), then turning around and showing that if \( f \) satisfies the appropriate restrictions, we can actually construct \( \mu \).

**4. The reals.** Let \( \mu \) be a Gaussian measure on \( \mathbb{R}^n \). Then \( \text{supp } \mu \) is a subspace of \( \mathbb{R}^n \); we may therefore assume \( \text{supp } \mu = \mathbb{R}^n \).
**Theorem 4.1.** There is a symmetric nonsingular complex \( n \times n \) matrix \( B' \) such that \( d\mu(x) = \exp(\pi Bx, x) dx. \) (Here, \( dx \) is Haar measure.) In particular, \( \mu \) is absolutely continuous.

The proof amounts to showing that \( \exp(-nx,x) \) is \( \mu \)-integrable for large enough \( n \). After that, a Fourier transform argument gives the rest.

Arguments similar to those of Theorem 4.1 can be used to find all Gaussian measures on \( \mathbb{R}^n \times C \), where \( C \) is an arbitrary compact group.

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**Bibliography**


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