1. Definition of sphere bundles. Let $M^n$ be an $n$-dimensional, $C^\infty$-manifold. Define $T(M)$ to be all vectors tangent to $M$ of unit length. Define $\varphi: T(M) \to M$ by $\varphi(\text{vector}) = \text{initial point of the vector}$. Then $\varphi$ is a continuous function with $\varphi^{-1}(m)$ homeomorphic to $S^{n-1}$ if $m \in M$. $(T(M), \varphi, M)$ is an example of an $(n-1)$-sphere bundle.

Let me now abstract some of the properties of this example and define an $(n-1)$-sphere bundle. An $(n-1)$-sphere bundle $\xi$ is a triple $(E, \varphi, X)$, where $\varphi: E \to X$ is a continuous function, $X$ has a covering by neighborhoods $\{V_a\}$ such that $h_a: \varphi^{-1}(V_a) \to V_a \times S^{n-1}$, where $h$ is a homeomorphism, $h_a(e) = (\varphi(e), S_a(e))$. That is, we can give coordinates to $\varphi^{-1}(V_a)$ using $V_a$ and $S^{n-1}$. Furthermore, there is a condition on changing coordinates; namely, if $e \in \varphi^{-1}(V_a \cap V_B)$, then $h_a(e) = (\varphi(e), S_a(e))$ and $h_B(e) = (\varphi(e), S_B(e))$ and we obtain a function $S^*: S^{n-1} \to S^{n-1}$ given by $S^*(S_a(e)) = S_B(e)$, defined for each $\varphi(e) \in V_a \cap V_B$. We demand that $S^* \in O(n)$, the orthogonal group of homeomorphisms of $S^{n-1}$. Finally, $S^*$ depends on $\varphi(e)$ and this dependence must be continuous.

Two $(n-1)$-sphere bundles $\xi$ and $\eta$ over $X$ are called equivalent if there is a homeomorphism $F: E_\xi \to E_\eta$ such that

$$
\begin{array}{ccc}
F & E_\xi & \to & E_\eta \\
\varphi \searrow & & \nearrow \varphi \\
& X & \\
\end{array}
$$

commutes and such that $F|\varphi^{-1}(x) \in O(n)$ for all coordinates on $\varphi^{-1}(x)$.

A very important example of an $(n-1)$-sphere bundle is the following one. Let $BO(n) =$ the Grassmann space of all $n$-planes through the origin in $\mathbb{R}^n$. Let $EO(n)$ be the set of pairs, an element of $BO(n)$ and a unit vector in that $n$-plane. Let $\varphi: EO(n) \to BO(n)$ be the first element of the pair. The importance of this example is shown by the following classification theorem.

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1 An address delivered before the New York meeting of the Society by invitation of the Committee to Select Hour Speakers, April 13, 1968; received by the editors April 21, 1969.

2 In order not to obscure the structure of the subject, I have left out a number of technicalities; in fact some of the statements may be incorrect as stated.
CLASSICAL CLASSIFICATION THEOREM. The equivalence classes of 
\((n-1)\)-sphere bundles over \(X\) are in one-to-one correspondence with the 
homotopy classes of maps of \(X\) into \(BO(n)\).

2. Definition of characteristic classes. Roughly speaking, a char­
acteristic class is a cohomology class in \(H^*(X)\) assigned to a bundle 
\(\xi\) over \(X\) which is natural with respect to bundle maps. Rather than 
give a precise definition, let me give a construction.

Let \(u \in H^*(BO(n))\). Let \(\xi\) be an \((n-1)\)-sphere bundle over \(X\) cor­
responding to a map \(f_\xi: X \to BO(n)\) by the above theorem. \(u\) defines 
a characteristic class \(u(\xi) \in H^*(X)\) by \(u(\xi) = f_\xi^*(u)\), where \(f_\xi^*:\n
H^*(BO(n)) \to H^*(X)\) is the homomorphism induced by \(f_\xi\) (recall that 
\(f_\xi^*\) depends only on the homotopy class of \(f_\xi\)).

Thus, to study characteristic classes, we must study \(H^*(BO(n))\).
The answers, with various fields for coefficients, are as follows: \(^3\)
\(H^*(BO(n); \mathbb{Z}_2)\) is a polynomial ring over \(\mathbb{Z}_2\) on generators \(W_i \in 
H^i(BO(n); \mathbb{Z}_2), i = 1, \ldots, n\). \(W_i(\xi) = f_\xi^*(W_i)\) is called the \(i\)th Stiefel-
Whitney class of \(\xi\). \(H^*(BO(\infty); \mathbb{Z}_p)\) and \(H^*(BO(\infty); \mathbb{Q})\) are poly­
nomial rings over \(\mathbb{Z}_p\) (\(p\) is an odd prime) and \(\mathbb{Q}\) respectively on gen­
erators \(P_i \in H^{4i}(BO(\infty); \mathbb{Z}_p)\) or \(H^{4i}(BO(\infty); \mathbb{Q}), i = 1, \ldots, P_i(\xi) = f_\xi^*(P_i)\) is called the \(i\)th Pontrjagin class of \(\xi\).

3. Some examples of applications of characteristic classes. The 
study of characteristic classes has been very useful in differential 
geometry, differential topology, and algebraic topology. I will now 
give a few examples of such applications.

I. Cobordism. Let \(M^n\) be a closed, connected, \(C^\infty\)-manifold of di­
mension \(n\). Then \(M^n = \partial W^{n+1}\), where \(W^{n+1}\) is a compact, connected, 
\(C^\infty\)-manifold with boundary, if and only if \(f_\tau^*: H^n(BO(n); \mathbb{Z}_2) \to H^n(M^n; \mathbb{Z}_2)\) is zero where \(\tau\) is the tangent bundle described at the 
beginning of this lecture [17].

II. Homotopy spheres. Let \(\Theta^n\) be the group of diffeomorphism 
classes of homotopy spheres. Pontrjagin classes have been used to 
study these groups. For example, \(\Theta^{16} \approx Z_{8128} [6]\).

III. Embeddings and immersions. Given \(M^n\), the problem is to find 
the smallest \(k\) such that \(M^n\) can be differentiably embedded or im­
mersed in \(R^{n+k}\). The initial results were proved using Stiefel-Whitney 
classes. The techniques now are quite complicated and we are now 
near to solving this problem for real and complex projective spaces.

IV. K-theory. Let \(KO(X)\) be the set of equivalence classes of 
bundles over \(X\), with dimension \(X < n\). This forms a group and acts

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\(^3\) An excellent introduction to the classical theory, including proofs of the follow­

ing assertions, is to be found in [10].
as a cohomology theory which has turned out to be very useful. For example, one can prove that the maximum number of linearly independent tangent vector fields on \( S^{n-1} \) is \( 2^e + 8d - 1 \), where \( n = (2q + 1)2^a \), \( b = c + 4d \), \( 0 \leq c \leq 3 \) [1].

4. More general bundles. In recent years it has become clear that one should study \((n-1)\)-sphere bundles where the changes of coordinates can be allowed to be in larger groups than \( O(n) \). Examples of such groups, in increasing size, are: \( \text{PL}(n) = \) piecewise linear homeomorphisms of \( S^{n-1} \), \( \text{Top}(n) = \) homeomorphisms of \( S^{n-1} \), and \( \text{G}(n) = \) homotopy equivalences of \( S^{n-1} \).

New classification theorem. Let \( H = \text{PL}, \text{Top}, \) or \( \text{G} \). There exists a space \( BH(n) \) such that the equivalence classes of \((n-1)\)-sphere bundles with group \( H \) over \( X \) are in one-to-one correspondence with the homotopy classes of maps of \( X \) into \( BH(n) \) ([11] and [14]).

Using this theorem, we can give the same construction of characteristic classes as we did in the classical case. In order to use these characteristic classes, we need to know \( H^*(BH(n)) \) with various coefficients. Most of the rest of this paper is devoted to describing what is known about \( H^*(BH) \), where \( BH = \lim_{n \to \infty} BH(n) \).

5. \( \pi_\ast(BH) \). Before stating the results on cohomology, let me first give the known results on the homotopy groups of the classifying spaces.

I. \( \pi_\ast(BO) \) is periodic of period 8 with \( \pi_{8k+i}(BO) = Z, Z_2, Z_2, 0, Z, 0, 0, 0, \) with \( i = 0, 1, 2, 3, 4, 5, 6, \) and 7 respectively [2].

II. \( 0 \to \pi_\ast(BO) \to \pi_\ast(BPL) \to \Gamma_{i-1} \to 0 \) is an exact sequence where \( \Gamma_{i-1} \) is a finite group which is partially known. Also, the structure of the exact sequence is known ([5] and [4]).

III. \( \pi_\ast(\text{Top}/\text{PL}) = 0 \) if \( i \neq 3 \), and \( \pi_3(\text{Top}/\text{PL}) = Z_2 \) [7].

IV. \( \pi_\ast(BG) = \pi_{i-k}(S^k) \), \( k \) large; hence, known to a certain extent.

V. \( \pi_\ast(G/\text{PL}) \) is periodic of period 4 with \( \pi_{4k+i}(G/\text{PL}) = Z, 0, Z_2, 0, \) with \( i = 0, 1, 2, \) and 3 respectively [15].

6. \( H^*(BH; Q) \). Using the above results on \( \pi_\ast \), it is easy to see that \( H^*(BG; Q) = 0 \) if \( i > 0 \) and that \( H^*(BTop; Q) \to H^*(BPL; Q) \to H^*(BO; Q) \to Q[P_1, \ldots] \) are all isomorphisms.

7. \( H^*(BH; Z_2) \). There exists a connected Hopf algebra \( C(H) \) over the mod 2 Steenrod algebra \( A_2 \) such that \( H^*(BH; Z_2) = H^*(BO; Z_2) \otimes C(H) \), as Hopf algebras over \( A_2 \) [3]. \( C(0) \) is trivial of course. \( C(G) \) is 2-connected, and its structure has been determined recently [9]. \( C(\text{PL}) \) and \( C(\text{Top}) \) are still unknown.
8. $H^*(BH; Z_\rho)$, $\rho$ odd. The situation is a little different from the case $\rho = 2$. Analogous to the case $\rho = 2$ we have $H^*(BG; Z_\rho) \approx (Z_\rho[q_i] \otimes E(\beta q_i)) \otimes C_\rho(G)$, where $q_i \in H^{i(2p-2)}(BG; Z_\rho)$ is the Wu class, $\beta q_i$ is its Bockstein, and $C_\rho(G)$ is a Hopf algebra over $A_\rho$ [13]. Furthermore, $C_\rho(G)$ is $(p(2p-2)-2)$-connected and its complete structure has been found very recently [7].

$H^*(B\text{Top}; Z_\rho) \rightarrow H^*(B\text{PL}; Z_\rho)$ is an isomorphism by §5, III, so we need only study $H^*(B\text{PL}; Z_\rho)$. It is an unpublished theorem that if $\rho$ is an odd prime, BPL is of the same mod $\rho$ homotopy type as $BO \times B \text{Coker } J$, where $B \text{Coker } J$ is a space whose homotopy groups are the cokernel of the homeomorphism $J: \pi_*(BO) \rightarrow \pi_*(BG)$ [16]. However, the map $BO \times pt. \rightarrow BO \times B \text{Coker } J \rightarrow B\text{PL}$ is not the usual map so this is quite different from the case $\rho = 2$. Also, the map $J_{\text{PL}}: B\text{PL} \rightarrow BG$ has the property that $J^{\rho*}_{\text{PL}}(\beta q_i) = 0$ if $i \leq \rho$ and is not zero if $i = \rho + 1$. The best conjecture at present seems to be that $C_\rho(G) \approx H^*(B \text{Coker } J; Z_\rho)$. To complete the picture, we need to know $J^{\rho*}_{\text{PL}}(q_i)$ explicitly [12].

9. **Applications.** One expects that a good knowledge of these new characteristic classes will lead to many applications as in the classical case. I mention only one, namely that §3, I generalizes to the PL case. That is, let $M^n$ be a closed, connected PL-manifold. Then $M^n = \partial W^{n+1}$, where $W^{n+1}$ is a compact, connected PL-manifold with boundary, if and only if $f^*_n: H^n(B\text{PL}; Z_\rho) \rightarrow H^n(M^n; Z_\rho)$ is zero [3]. A similar theorem is true for oriented $C^\infty$-manifolds, but for oriented PL-manifolds, it fails in dimension 27 (though true in lower dimensions) [12].

**Bibliography**


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