A THEOREM OF STOUT

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In a paper that appeared in Mathematische Annalen in 1968, E. L. Stout proved a generalization of a theorem of Rado which states that if \(f\) is continuous on the set \(D = \{z : |z| < 1\}\) and holomorphic on \(D - f^{-1}(0)\), then \(f\) is holomorphic on \(D\). Stout's generalization is as follows:

**Theorem.** Let \(E\) be a set of capacity zero in the complex plane, and let \(E_0\) be a relatively closed set in \(D\). If \(f\) is a function bounded and holomorphic in \(D - E_0\), if \(f\) does not vanish identically, and if for every sequence \(\{z_n\}\) in \(D - E_0\) such that \(z_n \to z_0 \in E_0\) and \(\lim f(z_n) = w_0\) exists it is the case that \(w_0 \in E\), then \(f\) is holomorphic throughout \(D\).

We shall generalize this theorem by replacing \(D\) by an arbitrary hyperbolic Riemann surface and replacing the assumption that \(f\) is bounded by the assumption that \(f\) belong to the Hardy class \(H^p\), \(0 < P \leq \infty\), i.e., that \(|f|^P\) possesses a harmonic majorant. Thus our theorem reads as follows:

**Theorem.** Let \(E\) be a set of capacity zero in the complex plane, and let \(E_0\) be a closed subset of a hyperbolic Riemann surface \(R\). If \(f \in H^p(R - E_0)\), i.e., \(f\) is of class \(H^p\) on each component of \(R - E_0\), \(0 < P \leq \infty\), if \(f\) is nonconstant on some component of \(R - E_0\), and if for every sequence \(\{a_n\}\) in \(R - E_0\) such that \(a_n \to a_0 \in E_0\) and \(\lim f(a_n) = b_0\) exists it is the case that \(b_0 \in E\), then \(f\) is holomorphic throughout \(D\).

**Remark.** In the proof we shall use the same notation and terminology in [1] and [2].

**Proof.** Let \(S = R - E_0\) and let \(S_0\) be a component of \(S\) on which \(f\) is nonconstant. It then follows that \(S_0\) is hyperbolic and that its universal covering surface is \(D\). Let \(\pi\) denote the projection map of \(D\) onto \(S_0\). Since \(\pi\) is a Fatou mapping, the fine limit function \(\hat{\pi}\) is defined a.e. on \(\partial D\). Here a.e. refers to Lebesgue measure. Since \(D\) is a regular covering surface of \(S_0\), it follows that every point \(P'\) in \(S_0\) has a neighborhood \(V\) with the property that each component of \(\pi^{-1}(V)\) is compact. Hence \(\pi\) is of Blaschke type. It follows that for a.e. point

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Let us now regard $\pi$ as a mapping of $D$ into $R$. $\pi$ is still a Fatou mapping and hence $\hat{\pi}$ is defined a.e. on $\partial D$. Consequently for a.e. $b \in \partial D$, $\hat{\pi}(b)$ is defined and either lies in $\Delta_1$, the minimal Martin boundary of $R$, or else lies in $E_0$. Since $f \in H^p(S_0)$, $f$ is a Fatou mapping and hence $f \circ \pi$ is a Fatou mapping. Hence $(f \circ \pi)^\wedge$ is defined a.e. on $\partial D$. If the set of points $b \in \partial D$ where $\hat{\pi}(b)$ is defined and lies in $E_0$, is of positive Lebesgue measure, then the set of points $f(\hat{\pi}(b)) = (f \circ \pi)^\wedge(b)$ would have to be of positive capacity, since $f$ is nonconstant, by a theorem of Doob; but on the other hand it would have to be of capacity zero since by hypothesis this set would have to be contained in $E$, and $E$ has capacity zero. Hence $\pi$ regarded as a mapping of $D$ into $R$ is of Blaschke type and hence by a theorem of Heins, $E_0$ is a set of capacity zero. Hence by a theorem of Parreau, $f$ has a holomorphic extension to $R$.

**Remark.** Actually, more is true. Thus $R - E_0$ is connected and the extended function is of class $H^p$ on $R$.

**References**


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