THREE-MANIFOLDS WITH FUNDAMENTAL GROUP A FREE PRODUCT

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1. Introduction. The purpose of this paper is to announce some results concerning the structure of a compact 3-manifold $M$ (possibly with boundary) where $\pi_1(M)$ is a free product. Related questions for $M$ closed have been considered in [1], [2], [4], [6], [8].

We use the term map to mean continuous function. If $M$ is a manifold, we use $\text{Bd} M$ and $\text{Int} M$ to stand for the boundary and interior of $M$, respectively. The disk $D$ is said to be properly embedded in the 3-manifold $M$ if

$$D \cap \text{Bd} M = \text{Bd} D.$$ 

The compact 3-manifold $H_n$ is called a handlebody of genus $n$ if $H_n$ is the regular neighborhood of a finite connected graph having Euler characteristic $1-n$.

The combinatorial terminology is consistent with that of [9]. The terms in group theory may be found in [3]. Furthermore, all maps and spaces are assumed to be in the PL category.

2. Bounded Kneser Conjecture.

THEOREM 2.1. Let $M$ denote a compact 3-manifold with nonvoid boundary where $\pi_1(M) \cong A \ast B$, a free product. Then there is a compact 3-manifold $M'$ with nonvoid boundary so that

(i) $M'$ has the same homotopy type as $M$, and

(ii) there is a disk $D'$ properly embedded in $M'$ where $M' - D'$ consists of two components $M_1$ and $M_2$ with $\pi_1(M_1) \cong A$ and $\pi_1(M_2) \cong B$.

OUTLINE OF PROOF. Let $K_A$ and $K_B$ denote CW-complexes with $\pi_1(K_A) = G$ and $\pi_n(K_A) = 0$, $n \geq 2$, $G = A$, $B$. Let $p$ denote a point not in $K_A \cup K_B$. Define $\overline{K_A}$ and $\overline{K_B}$ as the mapping cylinders of maps from $p$ into $K_A$ and $K_B$, respectively. Let $K$ denote the CW-complex obtained by identifying the copy of $p$ in $\overline{K_A}$ with the copy of $p$ in $\overline{K_B}$. It follows that $\pi_1(K) = A \ast B$ and $\pi_n(K) = 0$, $n \geq 2$ (see [1, p. 669]).

There is a simplicial map $f$ of $M$ into $K$ ($K$ may be chosen so that any finite collection of cells in $K$ has a simplicial subdivision) so that $f_*$ is an isomorphism of $\pi_1(M)$ onto $\pi_1(K)$.

LEMMA A. Let $M$, $K$, $f$, $p$ be as above. There is a map $g : M \to K$ so that

(i) $g$ is homotopic to $f$ relative to a base point of $\pi_1(M)$, and

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(ii) each component of $g^{-1}(p)$ is a disk properly embedded in $M$.

If $g^{-1}(p)$ has only one component, then we are finished. Otherwise, we continue in the following way.

For $M$, $K$, and $p$ as above, the map $g: M\to K$ is said to be reduced if each component of $g^{-1}(p)$ is a disk properly embedded in $M$. If $g$ is reduced, the complexity of $g$, $\#(g)$, is the number of components of $g^{-1}(p)$. Using this notation we wish to find a compact 3-manifold $M'$ having the same homotopy type as $M$ and a reduced map $g': M'\to K$ with $\#(g') = 1$ and $g'$ an isomorphism.

We are able to find a handlebody, $H_n$, of genus $n$ in $M$ and 3-cells $B_1, \cdots, B_k$ so that for each $i = 1, \cdots, k$,

$$B_i \cap H_n \subset \text{Bd} B_i \cap \text{Bd} H_n = A_i$$

is an annulus, $B_i \cap g^{-1}(p) = \emptyset$, $B_i \cap B_j = \emptyset$, $i \neq j$, and $M = H_n \cup B_1 \cup \cdots \cup B_k$. Under these conditions we say $(H_n, B_1, \cdots, B_k)$ is a Heegaard splitting for $M$ relative to $g$.

A path $\gamma$ in $\text{Bd} H_n$ is called a binding tie (see [7]) if $\gamma \cap g^{-1}(p)$ consists of the endpoints of $\gamma$, the endpoints of $\gamma$ are in distinct components of $g^{-1}(p)$, and the loop $g(\gamma)$ based at $p$ is contractible in $K$ (actually, since $g(\gamma) \subset \overline{K}_1$ or $\overline{K}_2$, $g(\gamma)$ is contractible in $\overline{K}_1$ or $\overline{K}_2$).

**Lemma B.** If $M$, $g$, $p$ are as in Lemma A and $\#(g) > 1$, then there is a Heegaard splitting $(H_n, B_1, \cdots, B_k)$ of $M$ relative to $g$ and a nonsingular (i.e. an arc) binding tie in $\text{Bd} H_n$.

We are now able to obtain a compact 3-manifold $M'$ having the same homotopy type as $M$ and a map

$$g': M' \to K$$

so that

(i) $g'$ is an isomorphism of $\pi_1(M')$ onto $\pi_1(K)$,
(ii) $g'$ is reduced,
(iii) $\#(g') = \#(g)$, and
(iv) there is a Heegaard splitting $(H_n', B_1', \cdots, B_k')$ of $M'$ relative to $g'$ and a nonsingular binding tie $\gamma'$ in $\text{Bd} H_n'$ so that $\gamma' \cap B_i' = \emptyset$ for each $i = 1, \cdots, k$.

**Lemma C.** If $M'$, $b'$, $\gamma'$, $p$ are as above, then there is a map $h': M' \to K$ so that

(i) $h'$ is homotopic to $g'$ relative to a base point for $\pi_1(M')$,
(ii) $h'$ is reduced, and
(iii) $\#(h') < \#(g')$. 

The following example shows that we cannot replace "homotopy type" by "homeomorphic" in part (i) of the previous theorem.

Let

\[ N = S^1 \times S^1 \times I \]

where \( S^1 \) is the 1-sphere and \( I \) the unit interval. Let

\[ T_0 = S^1 \times S^1 \times \{0\} \quad \text{and} \quad T_1 = S^1 \times S^1 \times \{1\}. \]

Choose disks \( D_0 \subseteq T_0 \) and \( D_1 \subseteq T_1 \). Let \( B \) denote the unit 3-cell. Choose disks \( D'_0 \) and \( D'_1 \) in \( \text{Bd} \, B \) so that \( D'_0 \cap D'_1 = \emptyset \). Let \( h \) denote an orientation reversing homeomorphism from the pair \((D'_0, D'_1)\) to \((D_0, D_1)\).

Define

\[ M = N \cup B, \]

the 3-manifold obtained from the disjoint union of \( N \) and \( B \) by identifying \( x \) with \( h(x), x \in D'_0 \cup D'_1 \).

It is an elementary exercise using Van Kampen's Theorem to see that

\[ \pi_1(M) \approx (Z \times Z) * Z \]

where \( Z \) is the infinite cyclic group. The 3-manifold \( M \) is irreducible; i.e. \( M \) is irreducible if each polyhedral two-sphere in \( M \) bounds a 3-cell in \( M \). Using this fact we obtain

**Proposition 2.2.** If \( M \) is the 3-manifold constructed above and \( D \) is a disk properly embedded in \( M \) so that \( M - D \) consists of two components \( M_1 \) and \( M_2 \), then the closure of either \( M_1 \) or \( M_2 \) is a 3-cell.

**Corollary 2.3.** If \( M \) is a compact 3-manifold and

\[ \pi_1(M) \approx A * B, \]

then there are compact 3-manifolds \( M_1 \) and \( M_2 \) so that \( \pi_1(M_1) = A \) and \( \pi_1(M_2) = B \).

### 3. Splitting along a disk.

**Theorem 3.1.** Let \( M \) denote a compact 3-manifold with nonvoid, connected boundary. Suppose that each disk \( D \) properly embedded in \( M \) separates \( M \). If

\[ \pi_1(M) \approx A * B, \]

then there is a disk \( D \) properly embedded in \( M \) so that \( M - D \) consists of two components \( M_1 \) and \( M_2 \) with \( \pi_1(M_1) = A \) and \( \pi_1(M_2) = B \).
OUTLINE OF PROOF. We proceed much along the same lines as in Theorem 2.1 through Lemma B of that theorem. Hence, borrowing from the notation of Theorem 2.1, suppose \( g: M \to K \) is a map so that \( g_* \) is an isomorphism and \( g \) is reduced. We assume \( g^{-1}(p) \) has more than one component (if \( g^{-1}(p) \) has only one component, then we are finished).

Let \( (H_n, B_1, \ldots, B_k) \) be a Heegaard splitting for \( M \) relative to \( g \). Suppose \( \gamma \) is a nonsingular binding-tie in \( \text{Bd} \ H_n \).

**Lemma D.** Using the above notation, there is a map \( h: M \to K \) so that

(i) \( h \) is homotopic to \( g \) relative to a base point for \( \pi_1(M) \),
(ii) \( h \) is reduced, and
(iii) \( \#(h) < \#(g) \).

**Theorem 3.2.** Let \( M \) denote a compact 3-manifold with nonvoid, connected boundary. If \( \pi_1(M) \cong \mathbb{Z} * G \), then there is a disk \( D \) properly embedded in \( M \) so that \( M - D \) is connected and \( \pi_1(M - D) \cong G \).

**Proof.** It is sufficient to show that there is a disk \( D \) properly embedded in \( M \) so that \( M - D \) is connected (see the remarks on p. 27, vol. II of [3]).

If each disk properly embedded in \( M \) separates \( M \), then by Theorem 3.1 there is a disk \( D \) properly embedded in \( M \) so that \( M - D \) has two components \( M_1 \) and \( M_2 \) with \( \pi_1(M_1) \cong \mathbb{Z} \) and \( \pi_1(M_2) \cong G \). However, we have

**Lemma E.** If \( M_1 \) is a compact 3-manifold with nonvoid boundary and \( \pi_1(M_1) \cong \mathbb{Z} \), then there is a disk \( D \) properly embedded in \( M_1 \) so that \( M_1 - D \) is connected.

We conclude in any case that the desired disk \( D \) may be obtained in \( M \).

A group \( G \) is said to be freely reduced if whenever \( G \cong G_1 * G_2 \) then neither \( G_1 \) nor \( G_2 \) is a free group.

**Theorem 3.3.** Let \( M \) denote a compact 3-manifold with nonvoid, connected boundary. Suppose \( \pi_1(M) \cong A * B \), a free product, and \( \pi_1(M) \) is freely reduced. Then there is a disk \( D \) properly embedded in \( M \) so that \( M - D \) consists of two components \( M_1 \) and \( M_2 \) with \( \pi_1(M_1) \cong A \) and \( \pi_1(M_2) \cong B \).

4. **Another proof of Kneser's Conjecture.** For an account of Kneser's Conjecture see [5] especially §§3, 12, 15, 17, and 20. Also, see [1], [2], [6]. Whitehead [8] was the first to obtain a satisfactory proof of Kneser's Conjecture.
The group \( G \) is said to be \textit{indecomposable} if whenever \( G \approx G_1 * G_2 \), then either \( G_1 \) or \( G_2 \) is the trivial group. If
\[
G \approx A_1 * \cdots * A_n
\]
where each \( A_i \) is indecomposable we call \( A_1 * \cdots * A_n \) a \textit{decomposition} of \( G \). Each finitely generated group has a decomposition which is unique up to order and isomorphism [3].

**Kneser Conjecture.** Let \( M \) denote a closed 3-manifold. Then
\[
\pi_1(M) \approx A * B,
\]
a free product, iff there is a polyhedral 2-sphere \( S \) in \( M \) so that \( M - S \) consists of two components \( M_1 \) and \( M_2 \) with \( \pi_1(M_1) \approx A \) and \( \pi_1(M_2) \approx B \).

We obtain Kneser's Conjecture as a corollary to

**Theorem 4.1.** Suppose \( M \) is a closed 3-manifold and
\[
\pi_1(M) \approx A_0 * A_1 * \cdots * A_k
\]
is a decomposition of \( \pi_1(M) \). Then there is a mutually exclusive collection of polyhedral 2-spheres \( S_1, \cdots, S_k \) in \( M \) so that \( M - U_i S_i \) consists of \( k + 1 \) components \( N_0, \cdots, N_k \) with \( \pi_1(N_i) \approx A_i \) for \( 0 \leq i \leq k \).

**Proof.** We proceed by induction on the integer \( k \) where \( k \geq 1 \). The general situation is no more difficult than the situation \( k = 1 \).

Let \( C \) denote a 3-cell in \( M \). Let \( M' = M - \text{Int} \, C \). Then \( \pi_1(M') \approx \pi_1(M) \). There are two cases to consider.

\textit{Case 1.} \( \pi_1(M) \) is not freely reduced.

We choose the notation in this case so that \( A_0 \approx Z \), the infinite cyclic group. Applying Theorem 3.2 we find a disk \( D \) properly embedded in \( M' \) so that \( M' - D \) is connected. Hence, we are able to find a 2-sphere \( S \) in \( M \) so that \( D \subset S \) and \( M - S \) is connected. We have

**Lemma F.** Let \( M \) denote a closed 3-manifold and suppose \( S \) is a 2-sphere in \( M \) so that \( M - S \) is connected. Then there is a 2-sphere \( S_1 \) in \( M \) so that \( M - S_1 \) consists of two components \( M_0 \) and \( \overline{M} \) where \( \pi_1(M_0) \approx Z \).

The inductive hypothesis applies to the 3-manifold obtained from \( \overline{M} \) by sewing a 3-cell \( B_1 \) onto \( S_1 \) along \( \text{Bd} \, B_1 \).

\textit{Case 2.} \( \pi_1(M) \) is freely reduced.

We proceed as in Case 1 only now we apply Theorem 3.3.

**Bibliography**


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