ON THE DECOMPOSITION OF MODULES

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Let \( R \) be a commutative ring with \( 1 \in R \), \( A \) and \( R \)-algebra—not necessarily commutative—and let \( M, N \) be two \( A \)-left-modules. We write \( N - \text{rk}(M) \geq s \), if \( M \cong sN \oplus M' \) for some \( A \)-left-module \( M' \) with \( s \cdot N \) short for \( N \oplus N \oplus \cdots \oplus N \), \( s \)-times.

Then one can prove the following generalization of a theorem of Serre (cf. [1] or [4]).

**Theorem 1. Assumptions.**

(i) \( N \) is finitely presented as \( A \)-left-module, \( \text{End}_A(N) \) finitely generated as \( R \)-module and \( M \) a direct summand in a direct sum of finitely presented \( A \)-modules;

(ii) the maximal ideal spectrum of \( R \) is noetherian of dimension \( d \);

(iii) for any maximal ideal \( m \) in \( R \) we have \( N_m - \text{rk}(M_m) \leq d + s \) with \( N_m \), resp. \( M_m \) the \( A_m = R_m \otimes_R A \)-module \( R_m \otimes_R N \), resp. \( R_m \otimes_R M \).

Then \( N - \text{rk}(M) \geq s \).

Moreover, if \( R \) is noetherian, \( \hat{R}_m \) the \( m \)-adic completion of \( R \) for some maximal ideal \( m \) in \( R \), resp. \( \hat{N}_m \), resp. \( \hat{M}_m \) the \( \hat{A}_m = \hat{R}_m \otimes_R A \)-module \( \hat{R}_m \otimes_R N \), resp. \( \hat{R}_m \otimes_R M \), then

\[
N_m - \text{rk}(M_m) \geq d + s \iff \hat{N}_m - \text{rk}(\hat{M}_m) \geq d + s.
\]

One can also prove the following generalization of the Cancellation Theorem of Bass (cf. [1]).

**Theorem 2. Assumptions.**

(i) and (ii) as in Theorem 1;

(iii) \( M \) contains a direct summand \( P \) with \( N - \text{rk}_m(P) > d \) for all maximal ideals \( m \) in \( R \), which is a direct summand in some \( s \cdot N \);

(iv) \( Q \) is an \( A \)-left-module, which is also a direct summand in some \( s \cdot N \), and \( M' \) is some \( A \)-left-module with \( Q \oplus M \cong Q \oplus M' \).

Then \( M \cong M' \).

The proof follows closely those of Serre and Bass [1], [4], once the following observations have been made:

(1) If \( N \) is any \( A \)-left-module and if \( B = \text{End}_A(N) \)—acting from the right on \( N \)—then the contravariant functor \( \text{Hom}_A(\cdot, N) \) from \( A \)-left-modules to \( B \)-right-modules defines a contravariant equivalence between the category \([N]\) of those \( A \)-left-modules \( P \), which are a direct summand in some \( s \cdot N \) (and all possible \( A \)-homomorphisms as morph-
isms) and the category of finitely generated projective $B$-right-modules. The functor $\text{Hom}_A(N, \cdot)$ defines thus an equivalence between $[N]$ and the category of finitely generated projective $B$-left-modules.

(2) If $N$ is a finitely presented $A$-module, $\rho: R \to \hat{R}$ a ring-homomorphism of $R$ into some commutative ring $\hat{R}$ (with $1 \in \hat{R}$ and $\rho(1) = 1$), such that $\hat{R}$ becomes a flat $R$-module, and if $\hat{M}$, resp. $\hat{N}$ stands for the $\hat{A} = \hat{R} \otimes_R A$-module $\hat{R} \otimes_R M$, then the natural homomorphism $\hat{R} \otimes_R \text{Hom}_A(M, N) \to \text{Hom}_\hat{A}(\hat{M}, \hat{N})$ is an isomorphism. (Cf. N. Bourbaki, Algèbre commutative, Chapter 1.)

(3) If $M, N$ are any two $A$-modules and $\phi: M \to N$ an $A$-homomorphism, define

$$P(\phi) = \{ \psi \in \text{End}_A(N) \mid \psi \in \text{Hom}_A(M, N) \},$$
$$I(\phi) = \{ \psi \in \text{End}_A(M) \mid \psi \in \text{Hom}_A(M, N) \},$$
$$P_0(\phi) = \{ r \in R \mid r \cdot \text{Id}_N \in P(\phi) \},$$
$$I_0(\phi) = \{ r \in R \mid r \cdot \text{Id}_M \in I(\phi) \}. $$

$P(\phi)$ is a right $\text{End}_A(N)$-ideal,

$I(\phi)$ is a left $\text{End}_A(M)$-ideal,

$P_0(\phi)$ and $I_0(\phi)$ are $R$-ideals.

The following statements follow easily from (2):

With $\rho: R \to \hat{R}$ as in (2) and $\hat{\phi} = \text{Id}_{\hat{R}} \otimes \phi: \hat{M} \to \hat{N}$ we have

If $N$ is finitely presented, then

$$P(\hat{\phi}) = \hat{P}(\phi), \quad P_0(\hat{\phi}) = \hat{P}_0(\phi).$$

If $M$ is finitely presented and $N$ a direct summand in a direct sum of finitely presented $A$-modules, then

$$I(\hat{\phi}) = \hat{I}(\phi), \quad I_0(\hat{\phi}) = \hat{I}_0(\phi).$$

There are many interesting applications of these observations and the two theorems above. For instance, in the case where $A$ is a separable order over a Dedekind ring $R$, all this specializes to something closely related to the results of Jacobinski [2], [3]. We mention some immediate consequences of (3): If $\rho: R \to \hat{R}$ is faithfully flat and $N$ finitely presented, then $\phi: M \to N$ is split-surjective if and only if $\hat{\phi}: \hat{M} \to \hat{N}$ is split-surjective.

If $N$ is finitely presented, then $\phi: M \to N$ is split-surjective if and only if $\phi_m: M_m \to N_m$ is split-surjective for all maximal ideals $m$ in $R$.

If $M$ is finitely presented and $N$ a direct summand in a direct sum of finitely presented modules, then $\phi: M \to N$ is split-injective if and only if all $\phi_m: M_m \to N_m$ are split-injective.
It is now not too difficult to go through with the proof of Serre [4], to get Theorem 1, whereas Theorem 2 is now a corollary to Bass’s Cancellation Theorem using (1).

REFERENCES