ON INTEGRAL REPRESENTATIONS

BY ANDREAS DRESS

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Let $G$ be a finite group and $p$ a prime. $G$ is called cyclic mod $p$ if there exists a normal $p$-subgroup $N \trianglelefteq G$ such that $G/N$ is cyclic.

Let $R$ be a commutative ring with $1 \in R$. Write $\mathbb{C}_R(G)$ for the set of subgroups $U \leq G$, which are cyclic mod $p$ for some appropriate prime $p (= p(U))$ with $pR \neq R$.

An $RG$-module $M$ is a finitely generated $R$-module, on which $G$ acts from the left by $R$-automorphisms. If $U \leq G$ we write $M|_U$ for the $RU$-module, one gets by restricting the action of $G$ on the $R$-module $M$ to $U$.

If $N$ is an $RU$-module, we write $N^U \rightarrow^g$ for the induced $RG$-module $RG \otimes_{RU} N$.

Two $RG$-modules $M, N$ are called weakly isomorphic, if there exists an $RG$-module $L$ and a natural number $k$, such that $k \cdot M \oplus L \cong k \cdot N \oplus L$ ($k \cdot M$ short for $M \oplus \cdots \oplus M$, $k$ times), we write then $M \sim N$.

REMARK. If the Krull-Schmidt-Theorem holds for $RG$-modules, we have

$$M \cong N \iff M \cong N.$$  

THEOREM 1. Let $M, N$ be two $RG$-modules. If $M|_U \cong N|_U$ for all $U \in \mathbb{C}_R(G)$, then $M \cong N$. Moreover there exist for any $U \in \mathbb{C}_R(G)$ two $R$-free $RG$-modules $M(U), N(U)$ with $M(U)|_V \cong N(U)|_V$ for all $V \leq G$, which do not contain any conjugate of $U$, but $M(U)|_U \cong N(U)|_U$.

One can get an even more precise statement by using Grothendieck-rings: Let $X(G, R)$ be the Grothendieck-ring of $RG$-modules with respect to split-exact sequences, i.e. $X(G, R)$ is an as additive group isomorphic to the free abelian group, generated by the isomorphism classes of $RG$-modules modulo the subgroup generated by all expressions of the form $M - M_1 - M_2$ with $M \cong M_1 \oplus M_2$—and the multiplication in $X(G, R)$ is given by the tensor-product $\otimes_R$ of $RG$-modules. Write $X_\mathcal{Q}(G, R)$ for $\mathcal{Q} \otimes X(G, R)$. Obviously $M \cong N$ if and only if $M$ and $N$ represent the same element in $X_\mathcal{Q}(G, R)$.

$X(\cdot, R)$ and $X_\mathcal{Q}(\cdot, R)$ are obviously contravariant functors from the category of groups into the category of commutative rings. Especially for $U \leq G$ one has restriction homomorphisms $\text{res}^U_V \colon X(G, R) \rightarrow X(U, R)$, $X_\mathcal{Q}(G, R) \rightarrow X_\mathcal{Q}(U, R)$ and Theorem 1 reads now

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THEOREM 1'. \( \prod_{U \in \mathcal{R}(G)} \text{res} |_U : X_\mathcal{O}(G, R) \to \prod_{U \in \mathcal{R}(G)} X_\mathcal{O}(U, R) \) is injective.

One can also describe the image of \( X_\mathcal{O}(G, R) \) in \( \prod_{U \in \mathcal{R}(G)} X_\mathcal{O}(U, R) \). More generally let \( \mathcal{U} \) be any family of subgroups of \( G \) closed with respect to subconjugation, i.e.

\[
U, V \leq G, \quad g \in G, \quad gVg^{-1} \leq U \in \mathcal{U}
\]

implies \( V \in \mathcal{U} \). For any such triple \( U, V \in \mathcal{U} \) and \( g \in G \) with \( gVg^{-1} \leq U \) one has a diagram:

\[
\begin{array}{ccc}
X_\mathcal{O}(G, R) & \xrightarrow{\phi} & X_\mathcal{O}(U, R) \\
\downarrow \tau_g & & \downarrow \\
X_\mathcal{O}(V, R) & \xleftarrow{\psi} & X_\mathcal{O}(U, R)
\end{array}
\]

the maps \( \phi \) and \( \psi \) given by restriction, the map \( \tau_g \) defined by \( V \to U, v \to gvg^{-1} \), and one can easily see, that this diagram is commutative. Thus \( \prod_{U \in \mathcal{U}} \text{res} |_U : X_\mathcal{O}(G, R) \to \prod_{U \in \mathcal{U}} X_\mathcal{O}(U, R) \) maps \( X_\mathcal{O}(G, R) \) into

\[
X_\mathcal{O}(\mathcal{U}, R) = \left\{ (x_U)_{U \in \mathcal{U}} \in \prod_{U \in \mathcal{U}} X_\mathcal{O}(U, R) \mid \tau_g(x_U) = x_V, \text{ whenever } U, V \in \mathcal{U}, g \in G \text{ and } gVg^{-1} \leq U \right\}
\]

and one has actually

THEOREM 2. The canonical map \( X_\mathcal{O}(G, R) \to X_\mathcal{O}(\mathcal{U}, R) \) is always epimorphic and is an isomorphism if and only if \( \mathcal{U} \subseteq \mathcal{R}(G) \).

It seems to be difficult, to prove a similar statement for \( X(G, R) \) instead of \( X_\mathcal{O}(G, R) \), but if \( X'(G, R) \) denotes the image of \( X(G, R) \) in \( X_\mathcal{O}(G, R) \), i.e. \( X(G, R) \) mod torsion, and if for any subconjugately closed family \( \mathcal{U} \) of subgroups of \( G \) we write \( \mathcal{U}' \) for \( \{ U \leq V, U \in \mathcal{U}, V/U \text{ a } p\text{-group} \} \) then one can prove

THEOREM 3. If \( (x_V)_{V \in \mathcal{U}} \subseteq X'(\mathcal{U}, R) \subseteq \prod_{V \in \mathcal{U}} X'(V, R) \) then the projection \( (x_U)_{U \in \mathcal{U}} \) of \( (x_V)_{V \in \mathcal{U}} \) into \( X'(\mathcal{U}, R) \subseteq \prod_{U \in \mathcal{U}} X'(U, R) \) is contained in the image of \( X'(G, R) \) in \( X'(\mathcal{U}, R) \).

REMARK. One can form a category \( \mathcal{U} \), whose objects are the subgroups in \( \mathcal{U} \) with \( \text{Hom}_\mathcal{U}(V, U) = \{ g \in G \mid gVg^{-1} \leq U \} \) and obvious composition. Then \( X(\cdot, R), X_\mathcal{O}(\cdot, R), X'(\cdot, R) \) are contravariant functors on \( \mathcal{U} \) and one has
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\[ X(U, R) = \text{proj} \lim_{U} X(\cdot, R), \quad X_{\mathcal{U}}(U, R) = \text{proj} \lim_{U} X_{\mathcal{U}}(\cdot, R), \]

\[ X'(U, R) = \text{proj} \lim_{U} X'(\cdot, R). \]

We will state one lemma, which is fundamental for the proof of the above theorems.

We say, that an \( RG \)-module \( M \) is weakly, resp. quasi-\( \mathcal{U} \)-induced, if there exists a natural number \( k \) and for any \( U \in \mathcal{U} \) two \( RU \)-modules \( N_{1}(U), N_{2}(U) \) such that

\[ k \cdot M \oplus \bigoplus_{U \in \mathcal{U}} N_{1}(U)^{U \leftarrow g} \cong \bigoplus_{U \in \mathcal{U}} N_{2}(U)^{U \leftarrow g}, \]

resp.

\[ k \cdot \left( M \oplus \bigoplus_{U \in \mathcal{U}} N_{1}(U)^{U \leftarrow g} \right) \cong k \cdot \left( \bigoplus_{U \in \mathcal{U}} N_{2}(U)^{U \leftarrow g} \right). \]

For a \( G \)-set \( S \) (i.e. a finite set, on which \( G \) acts from the left by permutations) we write \( R[S] \) for the free \( R \)-module, generated by \( S \), considered as \( RG \)-module by extending the action of \( G \) on the basis \( S \) linearly to \( R[S] \). Two \( G \)-sets \( S_{1}, S_{2} \) are \( \mathcal{U} \)-isomorphic (\( S_{1} \approx S_{2} \)), if the restrictions \( S_{1}|_{U} \) and \( S_{2}|_{U} \) to any \( U \in \mathcal{U} \) are isomorphic. Then we have the following

**Lemma.** For a group \( G \), a family \( \mathcal{U} \) of subgroups and a commutative ring \( R \) the following four statements are equivalent:

(i) the trivial \( RG \)-module \( R \) is weakly \( \mathcal{U} \)-induced;

(ii) any \( RG \)-module is weakly \( \mathcal{U} \)-induced;

(iii) \( X_{\mathcal{U}}(G, R) \rightarrow \bigoplus_{U \in \mathcal{U}} X_{\mathcal{U}}(U, R) \) is injective;

(iv) if \( S_{1}, S_{2} \) are two \( \mathcal{U} \)-isomorphic \( G \)-sets, then \( R[S_{1}] \approx R[S_{2}] \).

Any of these statements implies, that every \( RG \)-module is quasi-\( \mathcal{U} \)-induced with \( \mathcal{U} = \{ V \leq G \mid \text{there exists } g \in G, U \subseteq U \text{ with } gV^{-1} \subseteq U \} \), i.e. the subconjugate closure of \( \mathcal{U} \) and \( \mathcal{U} \) = \( \{ W \leq G \mid \text{there exists } V \subseteq \mathcal{U}, V \leq W, W/V \text{ a p-group} \} \) (just as before). Especially any \( RG \)-module is quasi-\( \mathcal{G}_{R}(G) \)-induced—which generalizes a well-known result of Brauer-Bermann-Witt-Swan for the case \( R = \mathbb{Q} \). In case \( \zeta \) is a \( \mathbb{Z} \)-root of unity with \( e = \exp(G) \) and there is a homomorphism \( \mathbb{Z}[\zeta] \rightarrow R \) one can sharpen this result, to generalize Brauer's result on elementary subgroups. Define \( \mathcal{G}_{R}(G) = \{ V \leq G \mid \text{there exists } N \leq V \text{ with } V/N \text{ elementary and } N \text{ a p-group for some } p \text{ with } pR \neq R \} \).

Then any \( RG \)-module is quasi-\( \mathcal{G}_{R}(G) \)-induced. One can also deduce intermediate statements, corresponding to the Bermann-Witt Theorem on \( K \)-elementary subgroups. There is still another way to generalize the above theorems: For any family \( \mathcal{U} \) of subgroups of \( G \),
define an $RG$-module $M$ to be $\mathfrak{U}$-projective, if $M$ is a direct summand in $\bigoplus_{\mathfrak{u} \in \mathfrak{U}} (M|_{\mathfrak{u}})^{\mathfrak{u} \rightarrow \mathfrak{g}}$. One can develop a theory of $\mathfrak{U}$-projective $RG$-modules completely analogous to D. G. Higman's theory in the case $\mathfrak{U} = \{ U \}$ and one can also define for any $RG$-module $M$ its family of vertices—corresponding to Green's theory, i.e. for any $RG$-module $M$ there exists a family of subgroups $\mathfrak{U}(M)$ such that $M$ is $\mathfrak{X}$-projective if and only if $\mathfrak{X} \supseteq \mathfrak{U}(M)$ ($\mathfrak{X}$ as before the subconjugate closure of $\mathfrak{X}$) and all subgroups in $\mathfrak{U}(M)$ are $p$-groups for various primes $p$ with $pR \neq R$. And one can prove that two $\mathfrak{U}$-projective $RG$-modules $M$ and $N$ are weakly isomorphic, if their restrictions $M|_V$ and $N|_V$ are weakly isomorphic for all $V \leq G$ which contain a normal $p$-subgroup $N \trianglelefteq V$ with $N \subseteq \mathfrak{U}$, $V/N$ cyclic and $pR \neq R$. In fact one proves Theorem 1 by using the above statement in some kind of complete induction argument, starting with $\mathfrak{U} = \{ E \}$, the trivial subgroup. There are corresponding generalizations of the other statements.

FREE UNIVERSITY OF BERLIN AND
INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540