ORDERS IN ARTINIAN RINGS

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The object of this note is to indicate some results concerning (right) skew polynomial rings over right orders in right Artinian rings. Detailed proofs will be published elsewhere.

We recall a few definitions first. A right Artinian ring is a ring with unity satisfying the descending chain condition on right ideals. A subring $S$ of a right Artinian ring $Q$ is called a right order in $Q$ if every regular element in $S$ is a unit in $Q$ and every $q \in Q$ can be expressed as $q = sc^{-1}$, where $s, c \in S$.

Let $R$ be a ring with unity and $S$ be a unitary subring of $R$. Suppose there exists an element $x \in R$ such that every nonzero $r \in R$ can be uniquely expressed as

$$r = \sum_{i=0}^{k} x^{n_i} s_i$$

where $s_i, 0 \leq i \leq k$, are nonzero elements of $S$ and $0 \leq n_0 < \cdots < n_k$ are integers. Further, suppose there exists a unitary monomorphism $\rho: S \rightarrow S$ such that $sx = xp(s)$ holds for every $s \in S$. In such a situation, we denote $R$ as $S[x, \rho]$ and call it a right skew polynomial ring over $S$. If $I$ is an ideal of $S$ such that $\rho(I) \subseteq I$ then $I$ is said to be $\rho$-invariant. We have

**Theorem 1.** Let $S$ be a ring with unity which is a right order in a right Artinian ring and let $R = S[x, \rho]$. Then $R$ is a right order in a right Artinian ring $\hat{Q}$. $\hat{Q}$ is semisimple if and only if $S$ is semiprime. $\hat{Q}$ is simple if and only if $S$ is semiprime and every nonzero $\rho$-invariant two-sided ideal of $S$ is an essential right ideal of $S$.

We need some more definitions. Let $Q$ be a semisimple (Artinian) ring, $\{f_1, \ldots, f_m\}$ be the set of all the distinct central idempotents of $Q$ which are primitive in the centre of $Q$ and let $\rho: Q \rightarrow Q$ be a monomorphism. Then there exists a unique permutation $\sigma$ on $\{1, \ldots, m\}$ such that $\rho(f_i) = f_{\sigma(i)}$ for $1 \leq i \leq m$. If $\sigma = \sigma_1 \cdots \sigma_k$ is a decomposition of $\sigma$ into disjoint cycles then $\max_{1 \leq i \leq m} \text{length } \sigma_i$ is called the shuffling index of $\rho$. Recall that a ring $R$ has right rank $m$ if $m$ is the least integer such that every right ideal of $R$ has a system of generators containing at most $m$ elements.
THEOREM 2. Let $Q$ be a semisimple right Artinian ring and $R = Q[x, \rho]$ where $\rho: Q \to Q$ is a monomorphism of shuffling index $m$. Then $R$ is a semiprime right hereditary right Noetherian ring of right rank $m$.

Further details about the structure of $R = Q[x, \rho]$ are obtained.

We briefly indicate the main steps in the proof of Theorem 1. Firstly, we obtain the following analogue of Faith-Utumi theorem [1] which may be of some independent interest.

THEOREM 3. Let $S$ be a right order in a right Artinian ring $Q$. Then there exists a set $\{e_i: 1 \leq i \leq n\}$ of orthogonal primitive idempotents in $Q$ with $\sum_{i=1}^{n} e_i = 1$ and there exist subgroups $M_{ij}$ of $e_iQe_j$, $1 \leq i, j \leq n$, such that $R = \bigoplus_{i,j=1}^{n} M_{ij}$ is a subring of $S$ and a right order in $Q$. Further, each $M_{ij}$ is a right order in the completely primary ring $e_iQe_j$.

It follows from Theorem 3 that, if $S$ is a right order in a right Artinian ring $Q$ and if $\rho: S \to S$ is a monomorphism then it can be uniquely extended to a monomorphism $\rho: Q \to Q$; further, if $Q[x, \rho]$ is a right order in a right Artinian ring $Q$ then $S[x, \rho]$ is also a right order in $Q$. It then remains to show that $Q[x, \rho]$ is a right order in a right Artinian ring for an arbitrary right Artinian ring $Q$.

We firstly consider the case when $Q$ is semisimple; in this case, if the shuffling index of $\rho$ is $m$ then $Q[x^m, \rho^m]$ is a semiprime principal right ideal ring and $Q[x, \rho]$ is a finitely generated right module over $Q[x^m, \rho^m]$. If $D$ denotes the set of all those right polynomials in $Q[x^m, \rho^m]$ which have a unit in $Q$ as a leading coefficient, then $D$ is an exhaustive right divisor set in $Q[x, \rho]$. Cf. [4].

For the case of an arbitrary right Artinian ring $Q$, we show that every monomorphism $\rho: Q \to Q$ induces a monomorphism $\bar{\rho}: \overline{Q} \to \overline{Q}$ where $\overline{Q} = Q/J(Q)$. The lift of the exhaustive right divisor set $D$ in $\overline{Q}[\hat{x}, \hat{\rho}]$ is then shown to be an exhaustive right divisor set in $Q[x, \rho]$. This establishes that $Q$ exists. Our arguments do not need nontrivial parts of the internal characterizations of right orders in semisimple rings or arbitrary right Artinian rings. Some special cases of Theorems 1 and 2 are known. Cf. [2], [3], [4].

REFERENCES


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