EMBEDDING SPHERES AND BALLS IN CODIMENSION $\leq 2$

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Communicated by William Browder, April 28, 1969

1. Introduction. In this note we announce some results on existence of PL embeddings of $n$-spheres and $n$-balls into a compact $(n-1)$-connected $q$-manifold $(n \geq q - 2)$ by extending techniques of our preceding papers [5], [4]. Details will appear later. The result for locally flat embeddings with codimension two is satisfactory, although in general the low dimensional cases are still open.

By $U_{k-1} D^n, U_{k-1} S^{n-1}$ we denote the disjoint unions of $r$ copies of the standard PL $n$-ball $D^n$, the standard PL $n$-sphere $S^n = \partial D^{n+1}$, resp. The embedding theorem of balls in codimension $\leq 2$ is as follows:

**THEOREM A.** Let $Q$ be a compact $(n-1)$-connected PL $q$-manifold with nonempty boundary $\partial Q$.

Let $\phi: U_{k-1} D^n \to Q$ be a map such that $\phi(U_{k-1} S^{n-1}) \subset \partial Q$ and $\phi|U_{k-1} S^{n-1}$ is a PL embedding.

(I). Suppose that one of the following holds.

(0) $q = n \neq 3, 4$,

(1) $q = n+1 \neq 4$,

(2) $q = n+2 \neq 4$ and $r = 1$.

Then $\phi$ is homotopic to a proper PL embedding $f: U_{k-1} D^n \to Q$ keeping $\phi|U_{k-1} S^{n-1}$ fixed.

(II). Suppose that $\phi|U_{k-1} S^{n-1}$ is locally flat, and that

(1) $q = n+1 \neq 4$ or

(2) $q = n+2 = \text{odd}$ and $r = 1$.

Then $\phi$ is homotopic to a locally flat PL embedding $f: U_{k-1} D^n \to Q$ keeping $\phi|U_{k-1} S^{n-1}$ fixed.

(Refer to [13, Chapter 8, Corollary 5].)

In case $q = n = 0$, Theorem A, (I) is equivalent to the generalized Poincaré conjecture. In case $q = n+1 = 4$, Theorem A is still open. In case $n = 2$ and $Q = D^4$, refer to [13, Chapter 8, Counterexample 1].

In case $q = n+2 = \text{even}$, Theorem A, (II) is false because of the existence of nonslice knots ([1] and [6, Chapter III]).

The embedding theorem of spheres in codimension $\leq 2$ is as follows:

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1 Work supported in part by Sakkokai Foundation and National Science Foundation grant GP-7952X.

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THEOREM B. Let $Q$ be a compact $(q-3)$-connected PL $q$-manifold. Suppose that $q \neq 4$.

(I). If $q \geq 5$, a basis of $H_{q-1}(Q; \mathbb{Z})$ can be represented by mutually disjoint locally flat PL$(q-1)$-spheres. In particular, any element of $H_{q-1}(Q; \mathbb{Z})$ can be represented by a locally flat PL$(q-1)$-sphere.

(II). Any element of $H_{q-2}(Q; \mathbb{Z})$ can be represented by a PL$(q-2)$-sphere.

(III). Further, if $q = \text{odd}$, then any elements of $H_{q-2}(Q; \mathbb{Z})$ can be represented by a locally flat PL$(q-2)$-spheres. (Refer to [2, Corollary 1.2].)

Theorem B, (I) is best possible by the homology reason. In case $q = 4$, Theorem B, (I) and (II) are still open.

Theorem B, (III) is best possible because of the following.

THEOREM C (WITH RONNIE LEE). For each even integer $n \geq 2$, there exists a compact PL$(n+2)$-manifold $Q$ which is an abstract regular neighborhood of $S^n$ such that no nontrivial element of $H_n(Q; \mathbb{Z})$ can be represented by a locally flat PL $n$-sphere. (Refer to [7].)

This is a modification of our preceding results [5, Theorem 2], whose proof may be improved to obtain the above by making use of Reidemeister torsions, which was pointed out to the author by Ronnie Lee.

As an implication of Theorems A and B we have the codimension $\leq 2$ extension of Irwin's Theorem [2].

THEOREM D. Let $M$ and $Q$ be compact PL $m$- and $q$-manifolds. Let $\phi: (M, \partial M) \to (Q, \partial Q)$ be a map such that $\phi|\partial M$ is a PL embedding and $\phi^{-1}(\partial Q) = M$.

Suppose that $m \geq 5$, $q-m \leq 2$ and

(1) $M$ is $(2m-q)$-connected, and

(2) $Q$ is $(2m-q+1)$-connected.

Then $\phi$ is homotopic to a proper PL embedding $f: M \to Q$ keeping $\partial Q$ fixed.

We remark here that by the normal PL bundle theory for locally flat PL embeddings [3], [11], [12] and the so-called Cairns-Hirsch smoothing theory [8], the adjective "locally flat PL" in theorems can be replaced by "smoothable," if $Q$ is smoothable.

2. The structure of compact $(q-3)$-connected $q$-manifolds $(q \geq 5)$. In the following, all things are considered from the piecewise linear viewpoint. Let $Q$ be a compact $(q-3)$-connected $q$-manifold with nonempty boundary $\partial Q$. Suppose that $q \geq 5$. Then by Poincaré-
Lefschetz duality and the universal coefficient theorem, $H_{q-1}(Q) \cong H^1(Q, \partial Q) \cong H_1(Q, \partial Q)$, $H_{q-2}(Q) \cong H^2(Q, \partial Q) \cong H_2(Q, \partial Q)$ are free of ranks $\alpha, \beta$, where $\alpha, \beta$ are the Betti numbers of $H_{q-1}(Q), H_{q-2}(Q)$, resp. Let $x = \{x_1, \ldots, x_\alpha\}$, $y = \{y_1, \ldots, y_\beta\}$ be given bases of $H_{q-1}(Q), H_{q-2}(Q)$ and let $\bar{x} = \{\bar{x}_1, \ldots, \bar{x}_\alpha\}$, $\bar{y} = \{\bar{y}_1, \ldots, \bar{y}_\beta\}$ be corresponding bases of $H_1(Q, \partial Q), H_2(Q, \partial Q)$ by the isomorphism above. From the general position we can represent these bases $\bar{x}$ and $\bar{y}$ by properly embedded arcs and disks having trivial normal bundles, since $Q$ is 1-connected and $H_2(Q, \partial Q) = H_1(\partial Q)$. Let $C$ be the complement of the union of open normal bundles of the disks and the arcs in $Q$. Then we have a handle decomposition; $Q = (\partial Q \times D) + (\tilde{\delta}_1) + \cdots + (\tilde{\delta}_a) + (\tilde{\gamma}_1) + \cdots + (\tilde{\gamma}_b) + C$, where handles $(\tilde{\delta}_i), (\tilde{\gamma}_i)$ are just the trivial normal bundles of the arcs, disks representing $x_i, y_k$ and hence of indices 1, 2, resp. By looking at this decomposition upside down, we have the dual decomposition;

$$Q = C + (\psi_1) + \cdots + (\psi_\beta) + (\phi_1) + \cdots + (\phi_\alpha),$$

where $(\phi_\alpha), (\psi_\beta)$ are the duals to $(\tilde{\delta}_a), (\tilde{\gamma}_b)$ and of indices $(q-1), (q-2)$, resp. Then the handles $(\phi_\alpha), (\psi_\beta)$ represent $j_*x, j_*y$, of $H_{q-1}(Q, C), H_{q-2}(Q, C)$, where $j_*: \tilde{H}_*(Q) \to H_*(Q, C)$ is the natural homomorphism from the reduced homology group $\tilde{H}_*(Q)$ to $H_*(Q, C)$. Notice that handles $(\phi_\alpha), (\psi_\beta)$ are mutually disjoint. Therefore, $H_*(Q, C)$ is torsion free and $j_*: \tilde{H}_*(Q) \to H_*(Q, C)$ is an isomorphism and $\tilde{H}_*(C) = 0$. On the other hand by the general position $C$ is 1-connected. Thus $C$ is a compact contractible $g$-manifold.

Now we have proved the following

**Theorem 2.1.** Let $Q$ be a compact $(q-3)$-connected $q$-manifold. Let $\alpha, \beta$ be the Betti numbers of $H_{q-1}(Q), H_{q-2}(Q)$. Suppose that $q \geq 5$. Given bases $x, y$ of $H_{q-1}(Q), H_{q-2}(Q)$, then we have a handle decomposition of $Q$ relative to a compact contractible $g$-manifold $C$;

$$Q = C + (\psi_1) + \cdots + (\psi_\beta) + (\phi_1) + \cdots + (\phi_\alpha)$$

such that handles $(\phi_\alpha), (\psi_\beta)$ are mutually disjoint, of indices $q-1, q-2$ and represent the bases $j_*x, j_*y$, resp.

**Remark.** For an $(n-1)$-submanifold $M$ of $\partial Q$, if $q = n+1$ and $\beta = 0$ or if $q = n+2$, then by the general position we may take the handle decomposition so that $M \subset \partial C$.

3. Embeddings of balls and spheres into a contractible manifold and a homology sphere. In codimension two case, the proof of Theorems A and B is based on the following special case of Theorem A
which is an extension of results on knot cobordisms due to Kervaire [6] and Levine [9].

**Theorem 3.1.** Let $C$ be a compact contractible $q$-manifold and let $\phi: \bigcup_{k=0}^{r} S^{n-1}_k \to \partial C$ be a locally flat embedding. Let $Q$ be a manifold obtained from $C$ by attaching $r$ handles of index $n$ via a framing of $\phi|_{U_k^{n-1} S^{n-1}_k}$.

Suppose that $q = n + 2 = 2m + 1$. Then $\phi|_{S^{n-1}_0}$ extends to a locally flat embedding $f$ such that $f(D^n)$ meets the right-hand ball with algebraic intersection number 1.

Suppose that $q = n + 2 = 2m + 2 \geq 6$. Then $f$ can be taken to be locally flat.

In codimension one case, it is based on the following

**Lemma 3.2.** Suppose that $q \geq 5$.

1. Let $C$ and $C'$ be compact contractible $q$-manifolds with homeomorphic boundaries $\partial C$ and $\partial C'$. Then a homeomorphism $h: \partial C \to \partial C'$ extends to a homeomorphism $H: C \to C'$.

2. A homology $(q-1)$-sphere $M$ bounds a contractible $q$-manifold.

This may be well known and implies the following

**Theorem 3.4.** Let $C$ be a compact contractible $q$-manifold and let $\phi: \bigcup_{k=0}^{r} S^{n-1}_k \to \partial C$ be an embedding. Suppose that $q = n + 1 \geq 5$.

Then $\phi$ extends to a proper embedding $f: \bigcup_{k=1}^{r} D^n_k \to C$.

Finally, the proof of Theorem A may be reduced to the locally flat case in virtue of the following

**Lemma 3.5.** Let $M$ be a homology $m$-sphere, and let $f: S^n \to M$ be an embedding. Suppose that $(m, n) \neq (4, 2)$. Then $f$ is isotopic to a locally flat embedding (perhaps by a locally knotted isotopy) keeping the complement of a given regular neighborhood of $f(S^n)$ in $M$ fixed.

This is a generalization of Fox-Milnor-Noguchi's Theorem [1], [10].

4. Applications: Some results on compact $(q-3)$-connected $q$-manifolds. An implication of Theorem 2.1 is the following generalization of [4, Theorem 3.11] and [5, Theorem 5].

**Theorem 4.1.** Let $Q$ be a compact $(q-3)$-connected $q$-manifold with nonempty boundary and let $\alpha, \beta$ be the Betti numbers of $H_{q-1}(Q), H_{q-4}(Q)$. Suppose that $q \geq 5$. 
Then the boundary \( \partial Q \) of \( Q \) has the homology of a manifold obtained from some copies of \( S^{r-1} \) and \( \beta \) copies of \( S^{r-2} \times S^1 \) by taking \( \alpha + 1 \) connected sums.

(2) Conversely, if a closed \((q-1)\)-manifold \( M \) has the homology of the manifold above, then \( M \) bounds a compact \((q-3)\)-connected \( q \)-manifold \( Q \) so that \( \alpha, \beta \) are the Betti numbers of \( H_{q-1}(Q) \) and \( H_{q-2}(Q) \).

(3) Moreover, if \((Q, \partial Q)\) is oriented, then there are at most \(2\beta\) distinct orientation preserving homeomorphism classes of oriented compact \((q-3)\)-connected \( q \)-manifold \((Q', \partial Q')\) whose boundaries \( \partial Q' \) are homeomorphic to \( \partial Q \) preserving orientations.

(4) In particular, if \( \beta = 0 \), then an orientation preserving homeomorphism \( h: \partial Q \to Q' \) extends to an orientation preserving homeomorphism \( H: Q \to Q' \).

Let \( Q \) be a compact \( q \)-manifold homotopy equivalent to \( S^n \). We define an invariant \( \omega(Q) \in \mathbb{Z} \) as follows: \( \omega(Q) = 0 \), if \( Q \) admits a locally flat embedding \( f: S^n \to Q \) which is a homotopy equivalence, and \( \omega(Q) = 1 \), otherwise. Note that \( \omega(Q) = 0 \) if and only if a basis of \( H_n(Q) \) can be represented by a locally flat \( n \)-sphere. In the situation above we have

**Theorem 4.2.** (I). The following statements are equivalent:

1. \( \omega(Q) = 0 \).
2. \( Q \) can be embedded in \( S^n \).
3. Any embedding \( f: S^n \to \text{Int } Q \) is isotopic to a locally flat embedding keeping the complement of a given regular neighborhood of \( f(S^n) \) in \( Q \) fixed.

(II). In particular, if \( \omega(Q) = 0 \), then \( Q \times D \) is homeomorphic to \( S^n \times D^{r-n+1} \) and the double of \( Q \) is homeomorphic to \( S^n \times S^{r-n} \).

The statements (I), (III) of Theorem B imply that \( \omega(Q) = 0 \), provided \( q = n + 1 \geq 5 \) or \( q = n + 2 = \text{odd} \).

**Corollary 4.3.** Let \( Q \) be a compact \( q \)-manifold homotopy equivalent to \( S^n \). Suppose that \( q \geq 5 \) and either \( q = n + 1 \) or \( q = n + 2 = \text{odd} \). Then all the statements of Theorem 4.2 hold.

In case \( q = 4 \), we have some weaker statements: Let \( Q \) be a compact \( 4 \)-manifold. Suppose that \( Q \) collapses a 2-subpolyhedron \( L \) homeomorphic to the wedge \( \bigvee_{k=1}^2 S^2_k \). We define an invariant \( I(Q) \) in \( \mathbb{Z}_2 \) as follows: \( I(Q) = 0 \), if each 2-sphere of \( L \) has the self-intersection number a multiple of 2 and \( I(Q) = 1 \), otherwise. Then we have

**Theorem 4.4.** (I). The following statements are equivalent:

1. \( I(Q) = 0 \).
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(2) $Q \times D$ is homeomorphic to a boundary connected sum of $\alpha$ copies of $S^3 \times D^3$.

(3) The double of $Q$ is homeomorphic to a connected sum of $\alpha$ copies of $S^3 \times S^3$.

(II). The following statements are equivalent:

1. $I(Q) = 1$.

2. $Q \times D$ is homeomorphic to a boundary connected sum of $\alpha - 1$ copies of $S^3 \times D^3$ and the nontrivial $D^3$ bundle over $S^3$.

3. The double of $Q$ is homeomorphic to a connected sum of $(\alpha - 1)$ copies of $S^3 \times S^3$ and the nontrivial $S^3$ bundle over $S^3$.

REFERENCES


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