UNIFORM ALGEBRAS ON CURVES

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1. Results. A recent result of H. S. Shapiro and A. L. Shields [4] states that if \( f \) and \( g \) are continuous complex valued functions on the unit interval \( I \) such that together they separate the points of \( I \) and also that \( f \) alone separates all but one pair of points, then the closed subalgebra of \( C(I) \) generated by \( f \) and \( g \) is all of \( C(I) \). Two generalizations are:

**Theorem.** Let \( A \) be a separating uniform algebra on \( I \) such that there exists an \( f \) in \( A \) which is locally 1-1, then \( A = C(I) \).

**Theorem.** Let \( A \) be a separating uniform algebra on \( I \) generated by two functions \( f \) and \( g \) such that there is a compact totally disconnected subset \( E \) of \( I \) such that

(i) \( f|E \) is constant, and

(ii) \( f \) separates every pair of points of \( I \) not both of which are in \( E \).

Then \( A = C(I) \).

The proofs use the notion of analytic structure in a maximal ideal space. J. Wermer first obtained results along these lines and further contributions were made by E. Bishop and H. Royden and then by G. Stolzenberg [5] who proved

**Stolzenberg's Theorem.** Let \( X \subseteq \mathbb{C}^n \) be a polynomially convex set. Let \( K \subseteq \mathbb{C}^n \) be a finite union of \( \mathcal{C}^1 \)-curves. Then \( (X \cup K)^* - X \cup K \) is a (possibly empty) pure 1-dimensional analytic subset of \( \mathbb{C}^n - X \cup K \). (See [5] for the notation and definitions.)

A further result of Stolzenberg (and Bishop) is that a \( \mathcal{C}^1 \) arc \( K \subseteq \mathbb{C}^n \) is polynomially convex and \( P(K) = C(K) \). It is well known that no smoothness is needed in \( \mathcal{C}^1 \) but that in higher dimensions further assumptions are required for the above conclusion. We have

**Theorem.** Let \( f_1, f_2, \ldots, f_n \in C(I) \) separate the points of \( I \) and suppose that for \( 1 \leq i \leq n-1 \), \( f_i \) is either \( \mathcal{C}^1 \) or real-valued. Then the separating uniform algebra which \( f_1, f_2, \ldots, f_n \) generate is \( C(I) \).

If all the \( f_i \), \( 1 \leq i \leq n-1 \) are real valued, this theorem reduces to a result of Rudin [3]; on the other hand, if we consider the image \( K \) of \( I \) under \( t \rightarrow (f_1(t), \ldots, f_n(t)) \) we obtain a generalization of Stolzenberg's result on smooth arcs.

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Applied to uniform algebras on the circle $T$, the methods of the previous theorems yield

**Theorem.** Let $A$ be a separating uniform algebra on $T$ which contains a function $f$ which is locally 1-1, then either

(i) $T$ is the maximal ideal space $M_A$ of $A$, in which case $A = C(T)$ or

(ii) $M_A - T$ is nonempty and has the structure of a 1-dimensional analytic space on which the functions in $A$ are analytic.

Finally we have the following which Shapiro and Shields [4] conjectured as an improvement of a result of Björk.

**Theorem.** Let $A = \{s: |z| < 1\}$. Let $F$ be a closed subset of $\Delta$ with $T \subseteq F \subseteq \Delta$ such that

(i) $F$ has no interior in $C$,

(ii) $\Delta - F$ is connected.

(iii) $(\Delta \cap F)^{-}$ does not contain $T$.

Let $g \in C(F)$ and suppose that the separating uniform algebra on $F$ generated by $g$ and $z$ is a proper subalgebra of $C(F)$. Then there exists $G \subseteq C(\Delta)$ such that

(i) $G|T = g|T$,

(ii) $G$ is analytic on $\Delta - F$.

The proofs [1] will appear elsewhere, together with more complete references to the literature. J. E. Björk [2] has independently obtained similar results.

**2. A special case.** In order to indicate the methods, we prove the following special case of the first mentioned theorem.

**Proposition 1.** Let $A$ be a separating uniform algebra on $I$ which contains a function $f$ which separates all but a finite number of pairs of points of $I$. Then $A = C(I)$.

**Proof (Sketch).** It is easily seen that there are a finite number of functions in $A$ which separate the points of $I$ and so we may assume that $A$ is finitely generated by $f_1 = f, f_2, \cdots, f_n$. Let $K$ be the homeomorphic image of $I$ under the map $t \rightarrow (f_1(t), \cdots, f_n(t))$. Then $K$ is an arc in $C^n$ and $z_1$ (the first coordinate function) separates all but a finite number of pairs of points of $K$. Our goal is to prove $P(K) = C(K)$. We note that $C - z_1(K)$ has finitely many components and in order to give a proof by induction on this number we prove a more general result.

**Proposition 2.** Let $K$ be a finite disjoint union of arcs in $C^n$. Suppose $z_1$ separates all but a finite number of pairs of points of $K$. Then $P(K) = C(K)$ and (hence) $K$ is polynomially convex.
Proof. Let \( L = z_1(K) \). \( C^1 - L \) has finitely many components. The proof will be by induction on this number \( k \).

\( k = 1 \): \( L \) does not separate the plane and \( L \) has no interior and so \( P(L) = C(L) \). It follows that \( z \mapsto z \) is in \( P(L) \) and so \( z \circ z_1 = z_1 \in P(K) \).

It is easily seen from the Stone-Weierstrass theorem that \( P(K) \) contains every \( f \in C(K) \) which identifies the points that \( z_1 \) does. From this it follows that \( P(K) = C(K) \).

Next we assume the result for \( k - 1 \) and prove it for \( k > 1 \). Assume, for the moment, that \( K \) has been proved to be polynomially convex. Then \( L_1 = z_1(K) \) is the spectrum of \( z_1 \) as an element of \( P(K) \).

As \( R(L) = C(L) \) it follows from the Gelfand theory that \( F \circ z_1 \in P(K) \) for all \( F \in C(L) \). In particular, \( z_1 \in P(K) \) and, as above, \( P(K) = C(K) \).

It remains to show \( K \) is polynomially convex. Suppose not. Let \( \Omega \) be a bounded component of \( C^1 - L \) such that there is an arc \( \gamma \subseteq \partial \Omega \) which is also in the boundary of \( \Omega_\infty \), the unbounded component of \( C^1 - L \). Let \( \gamma^0 \) denote \( \gamma \) with its endpoints deleted. We may assume \( z_1 \) is \( 1 \)-1 on \( z_1^{-1}(\gamma) \cap K \). Since \( \gamma \) is in the boundary of \( \Omega_\infty \), \( z_1^{-1}(\gamma) \cap K = z_1^{-1}(\gamma) \cap \hat{K} \) by [5]. Let \( K_1 = K - z_1^{-1}(\gamma^0) \). Then \( K_1 \) satisfies the hypotheses of our proposition for the case \( k - 1 \). So by the induction hypothesis, \( P(K_1) = C(K_1) \) and \( K_1 \) is polynomially convex. We claim that \( z_1(K) \cap \Omega \neq \emptyset \). In fact if \( p \in \hat{K} - K_1 \), as \( K_1 \) is polynomially convex, there is a polynomial \( f \) such that \( f(p) = 1 > ||f||_{x_1} \). Let \( T \) be the component of \( \{ q \in \hat{K} : ||f(q)|| \geq 1 \} \) which contains \( p \). Then by the local maximum modulus principle, \( T \) meets \( K \); hence \( T \) meets \( z_1^{-1}(\gamma^0) \cap K \). Hence \( z_1(T) \) meets \( \gamma \) and so clearly \( z_1(T) \) meets \( \Omega \).

Now by considering closed Jordan domains whose interiors are contained in \( \Omega \), whose boundaries contain \( \gamma \) and which meet \( \Omega \cap z_1(K) \), it follows by [5] that \( z_1^{-1}(\Omega) \cap \hat{K} \) is a 1-dimensional complex manifold in \( z_1^{-1}(\Omega) \) which is mapped by \( z_1 \) biholomorphically onto \( \Omega \).

Let \( \alpha \) be a straight line segment in \( \Omega \). Let \( \gamma_1 \) and \( \gamma_2 \) be arcs in \( \Omega \cup \{ \text{endpoints of } \gamma \} \) which join the endpoints of \( \alpha \) to those of \( \gamma \) such that \( \alpha \cup \gamma_1 \cup \gamma_2 \) is a Jordan curve bounding an open Jordan domain \( \omega \subseteq \Omega \). Let \( J = z_1^{-1}(\alpha) \cap \hat{K} \). \( J \) is a real analytic arc in \( C^\alpha \). Let \( X = (K - z_1^{-1}(\gamma)) \cup (z_1^{-1}(\gamma_1 \cup \gamma_2) \cap \hat{K}) \). Then \( X \) is polynomially convex as it is a union of arcs such that \( C^1 - z_1(X) \) has \( k - 1 \) components.

By Stolzenberg's theorem \( (X \cup J)^c - X \cup J \) is a 1-dimensional analytic subset of \( C^\alpha - X \cup J \). But by the local maximum modulus principle \( (X \cup J)^c = \hat{K} - z_1^{-1}(\omega \cup \gamma^0) \). It follows that \( \hat{K} - K \) is a 1-dimensional analytic subset of \( C^\alpha - K \). If \( \lambda \in \Omega \cap z_1(K) \), then \( z_1 - \lambda \) is an analytic function on \( \hat{K} \) which has a zero on \( \hat{K} \) and has a logarithm on \( \hat{K} \); this contradicts the argument principle [5]. We conclude that \( \hat{K} = K \). Q.E.D.
REFERENCES


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