This note states results extending those of Nash [2] on isometric embeddings of Riemannian manifolds in euclidean spaces; proofs and further details will be given elsewhere.

Let $M$ be a $d$-dimensional $C^\infty$ manifold. For convenience, we assume throughout that manifolds, whether compact or not, are connected. A metric on $M$ is defined to be a quadratic form on the tangent bundle of $M$; note that there is no assumption of nondegeneracy. We shall assume that all metrics are $C^\infty$. A Riemannian metric on $M$ is a metric whose restriction to the tangent space $T_q$ at a point $q \in M$ is positive definite, for all $q \in M$. A pseudo-Riemannian, or indefinite, metric is a metric whose restriction to the tangent space at each point is nondegenerate; if the nondegenerate restriction to $T_q$ has $n$ negative eigenvalues and $p$ positive eigenvalues, with $p + n = d$, the metric is said to have signature $(p, n)$ at $q$. The connectedness of $M$ implies that the signature is independent of the choice of $q \in M$.

$R^m$ will denote euclidean $m$-dimensional space, with the standard flat, positive definite metric, unless otherwise indicated; $R^n_p$ denotes euclidean $(n+p)$-dimensional space with flat metric of signature $(p, n)$. Thus $R^m_0 = R^m$. Let $F$ be a $C^\infty$ map, $F: M \to R^n_p$, and let $g$ be a metric on $M$; $F$ is said to be isometric for $g$ if $F^*(\cdot) = g$ where $^*$ denotes the metric for $R^n_p$ indicated above. Note that if $g$ is Riemannian and $F$ is isometric for $g$, then $F$ is necessarily an immersion and $n + p \geq d = \dim M$; for a general metric $g$, however, $F$ need not be an immersion. We shall concern ourselves with the question: given $M$ and a metric $g$ on $M$, for what $R^n_p$ do there exist isometric immersions, or isometric embeddings, $F: M \to R^n_p$?

1. A geometric argument for general metrics. Nash [2] guarantees the existence of isometric embeddings in some Riemannian euclidean space for any manifold with a Riemannian metric. The following argument reduces the general metric case to the Riemannian case, but requires higher dimension in the receiving euclidean space than necessary.

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ISOMETRIC EMBEDDINGS

PROPOSITION. Suppose any Riemannian metric on M has an isometric immersion in $\mathbb{R}^k$. Then any metric $g$ on M has an isometric immersion (embedding) in $\mathbb{R}_d^k(\mathbb{R}_{2d+1}^k)$.

OUTLINE OF PROOF. Using Whitney's standard results on immersions and embeddings, one obtains a "large" immersion or embedding $E: M \to \mathbb{R}^{2d}$ or $\mathbb{R}^{2d+1}$ such that $E^*(\cdot) + g$ is Riemannian. Then, if $F$ is isometric for $E^*(\cdot) + g$, $F \times E: M \to \mathbb{R}_d^k$ (or $\mathbb{R}_{2d+1}^k$) is isometric for $g$, where $F$ maps to the $k$ positive-eigenvalue coordinates and $E$ to the $2d$ (or $2d+1$) negative ones.

2. The compact case.

THEOREM 1. Let M be compact, d-dimensional, with metric $g$. There is an embedding $F: M \to \mathbb{R}_d^k$, $k = d(d + 5)/2$, which is isometric for $g$.

In [2], Nash shows that given a Riemannian metric $g$ on a compact manifold $M$, there is an isometric embedding of $M$ in euclidean $d(3d + 11)/2$-dimensional space, where the euclidean space is flat Riemannian; the above theorem reduces the dimension required for the euclidean space and extends the result to arbitrary $g$, but the euclidean space is pseudo-Riemannian, even if $g$ is Riemannian.

3. The noncompact case. The following Theorem 2 derives the existence of isometric immersions for noncompact manifolds from the isometric immersions of compact ones; the lemma, applicable to compact or noncompact manifolds, then provides isometric embeddings in the noncompact case. More specific results are given in Theorem 3.

THEOREM 2. Suppose that, for any metric $g$ on the $2d+1$-dimensional sphere $S^{2d+1}$, there is an isometric immersion of $S^{2d+1}$ in $\mathbb{R}^n$, where $\mathbb{R}^n$ has a flat Riemannian or pseudo-Riemannian metric. Then, if $M$ is any d-dimensional manifold with metric, there is an isometric immersion of $M$ in $\mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ has a flat Riemannian or pseudo-Riemannian metric. Furthermore, if, for any Riemannian metric $g$ on $S^{2d+1}$, there is an isometric immersion in $\mathbb{R}_d^n$, then any manifold of dimension $d$ with a Riemannian metric has an isometric immersion in $\mathbb{R}_d^{2n}$.

LEMMA. If a d-dimensional manifold $M$, compact or not, has an isometric immersion in $\mathbb{R}_d^n$ for any metric $g$ on $M$, then $M$ has an isometric embedding in $\mathbb{R}_d^{n+2d}$ for any $g$. Further, if, for every Riemannian metric $g$, there is an isometric immersion in $\mathbb{R}^n$, then there is an isometric embedding in $\mathbb{R}^{n+2d+1}$. 
THEOREM 3. Every manifold $M$ of dimension $d$, compact or not, has some isometric embedding in $\mathbb{R}^k$, $k = (2d + 1)(2d + 6)$, for every metric $g$ on $M$. If $g$ is a Riemannian metric, $M$ has an isometric embedding in $\mathbb{R}^k$, $k = (2d + 1)(6d + 14)$.

Theorem 3 improves Nash's result that a noncompact $d$-dimensional manifold $M$ with Riemannian metric has an isometric embedding in $\mathbb{R}^k$, $k = d(d + 1)(3d + 11)/2$.

4. The local case. If a metric $g$ is defined on an open set $U$ in a manifold $M$, then, given an open set $V$ with $\overline{V} \subset U$, there is a metric $\tilde{g}$ defined on $M$ such that $\tilde{g}|_V = g|_V$. Thus the global statements above imply corresponding local conclusions. However, in the local case, using A. Friedman's results [1] on the local analytic case, we can considerably improve the dimensional requirements for the cases of Riemannian and pseudo-Riemannian metrics.

THEOREM 4. Let $g$ be a metric defined on an open set in $\mathbb{R}^d$ and $u$ a point of $U$. Suppose that $g$ has signature $(p_1, n_1)$ at $u$, $p_1 + n_1 = d$. Then there is an open set $V$, with $u \in V$, such that there is an embedding of $V$ in $\mathbb{R}^n$, $n + p = d(d + 3)/2$, which is isometric for $g|_V$. If $g$ is Riemannian at $u$, then $n$ may be taken equal to 0. More generally, $n$ and $p$ may be chosen subject only to the restrictions $n \geq n_1$, $p \geq p_1$, $n + p = d(d + 3)/2$.

REFERENCES


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