COBORDISM OF REGULAR $O(n)$-MANIFOLDS

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A $C^\infty$ manifold $M$ together with a $C^\infty$ action of $O(n)$ on $M$ is said to be a regular $O(n)$-manifold if, for each $m \in M$, the isotropy group of $m$, $O(n)_m = \{g \in O(n) | gm = m\}$, is conjugate in $O(n)$ to $O(p)$ for some $p \leq n$; $O(p)$ is understood to be imbedded in $O(n)$ in the standard way [3]. Compact regular $O(n)$-manifolds $M_1$, $M_2$ are said to be (regularly) cobordant if there exists a compact regular $O(n)$-manifold $W^{s+1}$ with $\partial W^{s+1}$ equivariantly diffeomorphic to $M_1 \cup M_2$.

The set of cobordism classes of regular $O(n)$-manifolds of dimension $s$ will be denoted by $\pi O(n)_s$. $\pi O(n)_s$ is a graded algebra over $\pi_*$, the cobordism ring of unoriented manifolds; addition is given by disjoint union, multiplication by cartesian product (with the diagonal action $g(m_1, m_2) = (gm_1, gm_2)$, $(m_1, m_2) \in M_1 \times M_2$) and $\pi_*$ acts by cartesian product (with the obvious action $g(m_1, m_2) = (m_1, gm_2)$, $(m_1, m_2) \in M_1 \times M_2$, $[M_1] \in \pi_*$, $[M_2] \in \pi O(n)_s$).

EXAMPLES. (A) Let $M = \text{point}$. Then $[M] \in \pi O(n)_s$. The submodule of $\pi O(n)$ (as a $\pi_*$ module) generated by $[M]$ [i.e. trivial $O(n)$-manifolds] is isomorphic to $\pi_*$ and we clearly have a decomposition $\pi O(n)_s = \pi_* \oplus \pi O(n)_s$.

(B) Any manifold with a differentiable involution is a regular $O(1)$ manifold.

(C) If $M$ is a regular $O(n)$ manifold then by restricting the action to $O(n-1) \subset O(n)$ we get a regular $O(n-1)$ manifold. Since restriction respects cobordism there is an $\pi_*$ map $\rho: \pi O(n)_s \rightarrow \pi O(n-1)_s$.

(D) Given a regular $O(n)$ manifold $M$, one can extend the action to a regular $O(n+1)$ action on $O(n+1) \times O(n), M$ and hence there is an $\pi_*$ map $\text{ext}: \pi O(n)_s \rightarrow \pi O(n+1)_s$.

(E) Let $M$ be a regular $O(1)$ manifold and let $P$ be an $O(n-1)$ principal bundle. Then $P \times M$ is an $O(n-1) \times O(1)$ manifold and $O(n) \times O(n-1) \times O(1) P \times M$ is a regular $O(n)$ manifold. Hence, there is a homomorphism $h: \pi O(1) \otimes \pi_*(BO(n-1)) \rightarrow \pi O(n)_s$.

THEOREM. (i) $\pi O(n)_s$ is a free $\pi_*$ module on countably many generators:

(ii) the algebra structure is given by $xy = 0$ for $x, y \in \pi O(n)_s$, $n > 1$,

(iii) $\rho | \pi O(n)_s$ is the zero map,

(iv) $\text{ext} | \pi O(n)_s$ is a monomorphism onto a direct summand of $\pi O(n+1)_s$; $\text{ext} | \pi_*$ is zero,

(v) $h$ is an epimorphism.
COROLLARY 1. If $M$ is a nontrivial regular $O(n)$ manifold without boundary then there exists a regular $O(n)$ manifold $M'$, regularly cobordant to $M$, such that each isotropy group in $M'$ is either conjugate to $O(1)$ or is trivial. In particular, $SO(n)$ acts freely on $M'$.

COROLLARY 2. If $M$ and $M'$ are regularly cobordant $O(n)$-manifolds such that each isotropy group in $M$ and $M'$ is conjugate to $O(1)$ or is trivial then there is a regular cobordism $W$ between $M$ and $M'$ such that each isotropy group in $W$ is conjugate to $O(2)$ or $O(1)$ or is trivial.

Construction of generators. It is shown in [2] that $\mathfrak{N}O(1)_* = \sum_{i=2}^n \mathfrak{N}_*(BO(k))$. The isomorphism is constructed as follows: Let $E \to M$ be a differentiate $k$ plane bundle over $M$ and let $S(E)$, $P(E)$ be respectively the unit disc bundle, sphere bundle, and projective bundle of $E$. Let $L \to P(E)$ be the disc bundle associated to the $S^0$ bundle $S(E) \to P(E)$. Then $L \cup S(E) D(E)$ is the $O(1)$ manifold corresponding to $[E] \in \mathfrak{N}_*(BO(k))$. If $\xi_r$ denotes the canonical line bundle over $P_r$ then an $\mathfrak{N}_*$ basis for $\mathfrak{N}_*(BO(k))$ is given by the external products $\xi_{i_1} \times \xi_{i_2} \cdots \times \xi_{i_k}$ where $i_1 \geq i_2 \cdots \geq i_k \geq 0$ [2]. Similarly, every principle $O(n-1)$ bundle is cobordant to a linear combination (with coefficients in $\mathfrak{N}_*$) of bundles $P(s_1 \cdots s_{n-1}) = S^1 \times S^2 \cdots \times S^{n-1} \times Q(n-1) O(n-1)$ where $Q(n-1)$ is the product of $O(1)$ with itself $(n-1)$ times and $s_1 \geq s_2 \cdots \geq s_{n-1} \geq 0$. Hence, by example $E$ and $(v)$ of the theorem we have

Proposition 1. The manifolds

$M(i_1, i_2 \cdots i_k; s_1 \cdots s_{n-1}) = \bar{k}([\xi_{i_1} \times \xi_{i_2} \cdots \times \xi_{i_k}], [P(s_1 \cdots s_{n-1})])$

with $i_1 \geq i_2 \cdots \geq i_k; s_1 \geq s_2 \cdots \geq s_{n-1}$ and $k \geq 2$ generate $\mathfrak{N}O(n)_*$ as an $\mathfrak{N}_*$ module.

Note that the dimension of $M(i_1, \cdots i_k; s_1 \cdots s_{n-1})$ is

$$\sum_{j=1}^k (i_j + 1) + \sum_{j=1}^{n-1} s_i + \frac{n(n-1)}{2}.$$ 

These generators are not linearly independent—selecting a basis from them seems difficult. However, we do have the

Proposition 2. The collections of manifolds $M(i_1, \cdots i_{k-1}, 0; s_1 \cdots s_{n-1})$ $i_1 \geq i_2 \cdots \geq i_{k-1}; s_1 \geq s_2 \cdots s_{n-1}; k \geq 2$ are linearly independent over $\mathfrak{N}_*$ and generate a direct summand of $\mathfrak{N}O(n)_*$.

Proposition 3. All dependence relations among the generators are generated by relations involving a fixed $k$. 
The proof of these propositions involves an application of the spectral sequence of [4] for the group \(O(n)\) and the representation \(\rho_n \oplus \theta\) where \(\rho_n\) is the standard representation at \(O(n)\) and \(\theta\) is the trivial representation. In particular, we have

**Proposition 4.** There is a first quadrant spectral sequence \(E^r_{p,q}\) whose \(E^1\) term is given by

\[
E^1_{p,q} = \sum_k \mathcal{H}_* (BO(k) \times BO(q)) \quad 0 \leq q < n, \\
= \mathcal{H}_* (BO(n)) \quad q = n, \\
= 0 \quad q > n,
\]

and whose \(E^\infty\) term is associated to a filtration of \(\mathcal{H}(O(n)_\ast)\). Moreover, \(d_1: E^1_{p,q} \rightarrow E^1_{p,q+1}\) is given by \(d_1 = \rho_* \circ \pi_*\) where \(\pi_*: \mathcal{H}_* (BO(k) \times BO(q)) \rightarrow \mathcal{H}_* (BO(k-1) \times BO(1) \times BO(q))\) is the bordism transfer homomorphism [1] associated to the natural projection \(\pi: BO(k-1) \times O(1) \times O(q) \rightarrow BO(k) \times O(q)\) and \(\rho_*\) is induced by \(\rho: BO(k-1) \times O(1) \times O(q) \rightarrow BO(k-1) \times O(q+1)\).

The computations are best done in cobordism. One notes that \(d_1: \mathcal{H}_* (BO(k) \times O(q)) \rightarrow \mathcal{H}_* (BO(k+1) \times O(q-1))\) is linear as an \(\mathcal{H}_* (BO(q))\) module map. Let \(W_1 \cdots W_{k+q}\) be the cobordism Stiefel-Whitney classes of \(BO(k+q)\) and \(v_1, \cdots v_q\) the cobordism Stiefel-Whitney classes of \(BO(q)\).

**Proposition 5.** \(\mathcal{H}_* (BO(k) \times BO(q))\) is a free finitely generated \(\mathcal{H}_* (BO(q))\) module with generators \(\{v_1^i \cdots v_q^i\}\) where \(i_j \geq 0\) and \(\sum i_j \leq k\).

Finally, we have

**Proposition 6.** Up to units

\[
d_1(V_1^{i_1} V_2^{i_2} \cdots V_q^{i_q}) = 0 \quad \text{if} \quad \sum i_j < k, \\
= V_1^{j_1} V_2^{j_2} \cdots V_{q-1}^{j_{q-1}} \quad \text{if} \quad \sum i_j = k
\]

where \(V_i \in \mathcal{H}_* (BO(q-1))\). Hence, the sequence

\[
\mathcal{H}_* (BO(k) \times BO(q)) \xrightarrow{d_1} \mathcal{H}_* (BO(k+1) \times BO(q-1)) \rightarrow \mathcal{H}_* (BO(k + 2) \times BO(q - 2))
\]

is exact if \(k \geq 0\) and the spectral sequence collapses at the \(E^2\) level.

Theorem 1 now follows quickly.
REFERENCES


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