CONTRACTIVE PROJECTIONS AND PREDICTION OPERATORS

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1. Introduction. The purpose of this note is to present some results on characterizations of subspaces of a general class of Banach function spaces (BFS) admitting contractive projections onto them, and to include an application to nonlinear prediction (and approximation) theory.

Let $L^p$ be the subspace of all measurable scalar functions $f$ on $(\Omega, \Sigma, \mu)$ with $\rho(f) = \rho(|f|) < \infty$, where $\rho(\cdot)$ is a function norm, i.e., a norm with the additional properties

(i) $0 \leq f_n \uparrow \Rightarrow \rho(f_n) \uparrow$, and

(ii) $\rho(\cdot)$ verifies the triangle inequality for infinite sums. Then $L^p$ is also complete, called a BFS, (cf. [6] and [4]). It will also be assumed, for convenience, that $0 \leq f_n \uparrow \Rightarrow \rho(f_n) \uparrow \rho(f)$, the Fatou property. $\rho(\cdot)$ is an absolutely continuous norm (a.c.n.) if for each $f \in L^p$, $\rho(f_{\chi A_n}) \downarrow 0$ for any $A_n \in \Sigma$, $A_n \downarrow \emptyset$. If $\mathcal{X}$ is a $\mathcal{B}$-space, $L^p_{\mathcal{X}}$ is the space of $\mathcal{X}$-valued strongly measurable functions $f$ on $\Omega$, with $\rho(|f|_{\mathcal{X}}) < \infty$, where $\rho(\cdot)$ is as above. Then $L^p_{\mathcal{X}}$ is also complete. Finally let $\mathcal{M}_\mathcal{X} = \mathcal{S}\{f\chi : f \in L^p, \ x \in \mathcal{X}\} \subset L^p_{\mathcal{X}}$. A projection is a linear idempotent operator.

The projection problem, stated at the outset, has been first treated for $L^p = L^1$ in [5], and a more detailed consideration of the same case, with $\mu(\Omega) < \infty$, has been given in [2]. If $L^p = L^p$, also with $\mu(\Omega) < \infty$, it was then considered in [1], and these results were extended for $L^p = L^p$, the Orlicz spaces, with a.c.n. and $\mu$ $\sigma$-finite, in [10]. The general solution of the problem in the scalar case, and a less general one in the vector case, will be given below.

2. Contractive projections. Let $S \subset L^p$ be a closed subspace. If $L^p \neq L^p$, then, as is well known, not every $S$ is the range of a bounded projection. The positive solution is given by the following result for $L^p$-spaces. (An operator $T$ is positive if $Tf \geq 0$ for $f \geq 0$.)

Theorem 1. If $(\Omega, \Sigma, \mu)$ is a measure space, let $L^p(\Sigma)$ be the BFS defined above. Consider the statements:

(a) $S$ is the range of a (positive) contractive projection in $L^p(\Sigma)$.

(b) there is an isometric isomorphism $\Psi : L^p(\Sigma) \rightarrow L^p(\Sigma)$, ($\Psi =$ identity) such that

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(i) $\Psi(S)$ is a B-lattice, i.e., a selfadjoint space with real functions forming a lattice, and

(ii) $0 \leq f_n \in \Psi(S), f_n \uparrow f, f \in L^p(\Sigma) \Rightarrow f \in \Psi(S)$.

(c) there is a (positive) isometric isomorphism between some $L^p(\mathcal{A})$

on some measure space $(S, \mathcal{A}, \mu)$ and $S$.

(d) same as (c) except "topological equivalence" replaces "isometric isomorphism."

Then one has $(c) \Rightarrow (a) \iff (b) \Rightarrow (d)$. In case $\rho(\cdot)$ also verifies, $\chi_A \in L^p(\Sigma)$

for each $A \in \Sigma$ with $\mu(A) < \infty$, then $(a) \iff (c)$ also holds.

Remark. If $\rho(f) = \int_0^1 |f| \, d\mu/x$, with $\Omega = [0, 1]$, $\mu = \text{Leb. meas.}$, then $\rho(\cdot)$ is a function norm, but $\rho(\chi_0) = \infty$. Thus the last condition of the theorem is a restriction on $\rho$. It can be shown easily that $b(ii)$ automatically holds if $\rho$ is an a.c.n., but will be needed otherwise.

This result is proved through several isomorphisms using equivalent measure spaces and the results of [13]. However, for an application of the latter, a first reduction is needed and is provided by the following result which has independent interest.

Theorem 2. If $L^p(\Sigma)$ is a BFS on $(\Omega, \Sigma, \mu)$, then there exists a measure space $(S, \mathcal{A}, \nu)$ where $S$ is a locally compact space, $\mathcal{A}$ is a $\sigma$-field generated by the compact subsets of $S$ and $\nu$ is a measure assigning finite measure for compacts, in terms of which $L^p(S, \mathcal{A}, \nu)$, or $L^p(\mathcal{A})$, is isometrically (and lattice) isomorphic to $L^p(\Sigma)$. Moreover each $f$ in $L^p(\mathcal{A})$ has a compact support. If there exists a strictly positive element in $L^p(\mathcal{A})$, then $S$ can be chosen compact, so that $(S, \mathcal{A}, \nu)$ is a finite measure space.

If $\mu$ is $\sigma$-finite then a strictly positive element always exists in $L^p(\Sigma)$ (e.g., a weak unit, cf. [6, p. 153]) and the last part contains this case. This result is proved using a method of proof of ([8, Theorem 2.1]) and some results of [13]. (See also [3] for the $L^1$-case.) With this reduction, the problem of Theorem 1 can be transferred to $L^p(\mathcal{A})$. Then it can be isometrically embedded in $L^p(\widehat{A})$ on a localizable measure space $(\widehat{S}, \widehat{\mathcal{A}}, \widehat{\nu})$ where $\mathcal{A}$ goes, under an algebraic isomorphism, into a subring of $\widehat{\mathcal{A}}, [13, \text{Theorem 3.4}]$. Then the proof is successively reduced to the case of finite measure space where the methods and ideas of [2] and [10] can be generalized and used. In this way the full result of Theorem 1 is established.

In general there will be many contractive projections onto $S$, when one exists. The following gives a uniqueness result.

Proposition 3. Suppose $L^p(\Sigma)$ is a rotund (= strictly convex) and smooth (= norm is Gâteaux differentiable) reflexive space on $(\Omega, \Sigma, \mu)$. Then a closed subspace $S \subset L^p(\Sigma)$ can be the range of at most one contract-
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tive projection. If in particular $\mathcal{S} = L^p(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{F}$, a $\sigma$-field, then there exists a unique positive contractive projection onto $\mathcal{S}$, namely the (generalized) conditional expectation $E^\mathcal{B} : L^p(\mathcal{F}) \to L^p(\mathcal{B})$.

The case of $L^p = L^p$, $1 < p < \infty$, $\mu(\Omega) < \infty$, of the above result was obtained in ([1, p. 392]). The general form of $P$ is not-simple. The following case is illustrative.

**Proposition 4.** Let $P : L^p(\Sigma) \to L^p(\mathcal{B})$ be a contractive projection (which exists by Theorem 1), where $\mathcal{B} \subseteq \mathcal{F}$ is a $\sigma$-field with $\mu_\mathcal{B}$ $\sigma$-finite, and $L^p(\Sigma)$ is a BFS. Then there exists a locally integrable function $g$ such that

(i) $P(\cdot) = E^\mathcal{B}(g \cdot)$, and

(ii) $E^\mathcal{B}(g) = 1$ a.e., where $E^\mathcal{B}$ is the conditional expectation relative to $\mathcal{B}$.

This shows that while $E^\mathcal{B}$ itself is a contractive projection onto $L^p(\mathcal{B})$, it is not the general form of the operator. If $\rho$ is an a.c.n., then it can be shown that $g = 1$ a.e. here, and this is not necessarily true in the general case. The above two results are proved by an extension of the methods of [10]. A special case of the above proposition for $L^p$-spaces, with $\mu(\Omega) < \infty$, was discussed in [11].

For the case of $\mathcal{M}_G^p$ spaces, the following result holds.

**Theorem 5.** Let $L^p(\Sigma)$ and $\mathcal{M}^p_G$ be as defined in §1. If $\mathcal{S} \subseteq L^p(\Sigma)$ is a closed subspace, let $\mathcal{S} \mathcal{X} = \{ f \mathcal{X} : f \in \mathcal{S} \}$, $\mathcal{X} \subseteq \mathcal{M}^p_G$. Also let $\chi_A \in L^p(\Sigma)$ for each $A \subseteq \Sigma$ with $\mu(A) < \infty$. Then the following four statements are equivalent:

(i) $\exists$ contractive projection $P : L^p(\Sigma) \to \mathcal{S}$.

(ii) $\exists$ contractive projection $P : \mathcal{M}^p_G \to \mathcal{S}$.

(iii) $\exists L^p(\mathcal{B}_1, \mu_1)$, on some measure space $(\Sigma_1, \mathcal{B}_1, \mu_1)$ and $\mathcal{S}$ is isometrically isomorphic to $L^p(\mathcal{B}_1, \mu_1)$.

(iv) $\mathcal{S} \mathcal{X}$ is isometrically isomorphic to $\mathcal{M}^p_G(\mathcal{B}_1, \mu_1)$.

This result is proved on using Theorem 1, and the fact that $L^p \otimes \mathcal{X} \subseteq \mathcal{M}^p_G$ and is dense in the latter (see [9]). Here $\otimes$, is the greatest cross-norm, and one then uses a result on projections in cross-spaces [12, p. 58]. The general case of $L^p_\mathcal{X}$ itself does not seem to follow in this way. The above one already includes the $L^p_\mathcal{X}$, $1 \leq p \leq \infty$ case.

3. Prediction operators. A subspace $M \subseteq L^p$ is said to be a *Tshebyshev subspace* if for each $x \in L^p$ there is a unique $x_0 \in M$ with $\rho(x - x_0) = \min \{ \rho(x - y) : y \in M \}$. The operator $P_M : x \mapsto x_0 \in M$, is
called a *prediction operator* in nonlinear prediction theory. Though $P_M^* = P_M$, it is not linear in general. If it is linear, the powerful methods of linear analysis will be available in their study. So this is a natural question to treat. If $P_M$ is linear, then $Q = I - P_M$ is a contractive projection with $M$ as its null space (and the converse also holds). This is the connection between projections and predictions, and a solution can be presented as follows.

**Theorem 6.** Let $M \subseteq L^p$ be a Tshebyshev subspace, and $P_M$ be the prediction operator for $M$. If $P_M$ is linear then the quotient space $L^p/M$ is topologically equivalent to $L^p(\mathfrak{B})$ on some measure space $(S, \mathfrak{B}, \mu)$. Conversely, if $L^p/M$ is isometrically isomorphic to $L^p(\mathfrak{B})$ on some $(S, \mathfrak{B}, \mu)$ then $P_M$ is linear.

In case $\chi_A \subseteq L^p$ for each $A \subseteq \Sigma$, $\mu(A) < \infty$, then the above can be stated as: $P_M$ is linear $\iff L^p/M$ is isometrically isomorphic to an $L^p(\mathfrak{B})$. If $L^p = L^p$, $1 < p < \infty$, $\mu(\Omega) < \infty$, the latter has been obtained in [1]. The general case can be proved quickly with the results of the preceding section. However, it was noted in [10], that for the case $L^p \neq L^2$, $M$ must be relatively complicated since $P_M$ will not be linear if $M$ is of the form $L^p(\Sigma_1)$, $\Sigma_1 \subseteq \Sigma$, a $\sigma$-field.

The proofs of all the results above involve first a characterization of the adjoint space $(L^p)^*$ of $L^p$. This is involved. It is obtained by generalizing the work of ([7] and [4]) appropriately. With these results (and those of [9]), and of [13], the above bare sketch is completed. The details and related results will be published separately.

**References**


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