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HOLOMORPHIC MAPPINGS INTO TIGHT MANIFOLDS

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This paper gives an extension (Proposition 3) of Theorem C of H. Wu's paper [4], as well as a few other results. The terminology will be that of [4].

If $M$ and $N$ are complex manifolds $A(M, N)$ will denote the set of holomorphic mappings between $M$ and $N$. It is a topological space under the topology of uniform convergence on compact subsets of $M$. If $f_i$ is a sequence in $A(M, N)$ and $g$ is in $A(M, N)$ then $f_i \to g$ will mean that the $f_i$'s converge to $g$ in this topology. A pair $(N, d)$, where $N$ is a complex manifold and $d$ is a distance on $N$, will be called tight iff $A(M, N)$ is equicontinuous with respect to $d$ for all complex manifolds $M$. In fact $(N, d)$ is tight iff $A(B^n, N)$ is equicontinuous with respect to $d$, where $B^n$ here denotes the unit ball in $C^n$. For details see Part I of [4].

Our basic lemma, interesting for its own sake, is

**Proposition 1.** Let $M$ be a connected complex manifold, $U$ an open subset of $M$, and $(N, d)$ be tight. For $f \in A(M, N)$ define $i_U(f) \in A(U, N)$ to be the restriction of $f$ to $U$. Then $i_U$ is a homeomorphism of $A(M, N)$ into $A(U, N)$.

**Proof.** $i_U$ is one-to-one because $U$ is open in $M$. If $f_i \to g$ in $A(M, N)$ it is clear that $i_U(f_i) \to i_U(g)$. Thus $i_U$ is continuous, and it remains only to show that $i_U(f_i) \to i_U(g)$ in $A(U, N)$ implies that $f_i \to g$ in $A(M, N)$.

Suppose $i_U(f_i) \to i_U(g)$ in $A(U, N)$. Let $\mathcal{U} = \{V \subset M: V \text{ open in } M \}$ and $i_V(f_i) \to i_V(g)$ in $A(V, N)$. Partially order $\mathcal{U}$ by inclusion. If $V_1 \subset V_2 \subset V_3 \subset \cdots$ is a totally ordered chain in $\mathcal{U}$, it is clear that $V = \bigcup V_j$ is a member of $\mathcal{U}$. Since $U \subset \mathcal{U}$, $\mathcal{U}$ is not empty, so Zorn's Lemma implies that $\mathcal{U}$ contains maximal elements. Let $U_0$ be one such. We will show that $U_0 = M$.

If not, $\partial U_0 = \overline{U_0} - U_0$ is not empty. Let $x \in \partial U_0$ and $\epsilon > 0$. Since $N$
is tight there is a neighborhood, \( V \), of \( x \) such that \( y \in V \) implies \( d(h(x), h(y)) < \epsilon/3 \) for all holomorphic mappings \( h: M \to N \). Pick such a \( y \in V \cap U_0 \) and pick \( i_0 \) such that \( i > i_0 \) implies \( d(f_i(y), g(y)) < \epsilon/3 \). Then \( i > i_0 \) implies that
\[
d(f_i(x), g(x)) \leq d(f_i(x), f_i(y)) + d(f_i(y), g(y)) + d(g(y), g(x)) < \epsilon.
\]
This shows that \( f_i(x) \to g(x) \).

Let \( B \) be a taut (see [4, p. 199]) neighborhood of \( g(x) \) in \( N \). Since \( M \) is tight and \( f_i(x) \to g(x) \) there is a connected neighborhood, \( W \), of \( x \) in \( M \) such that \( f_i(W) \subset B \) for large \( i \). Now the set of holomorphic mappings from \( W \) to \( B \), \( A(W, B) \), is a normal family [4, p. 197]. If \( \{i(j)\} \) is any subsequence if \( Z^+ \) then \( f_i(W) \to g(x) \) and it follows that there is a subsequence \( \{j(s)\} \) of \( Z^+ \) such that \( i_W(f_{i(j(s)}) \to h \), where \( h \) is a member of \( A(W, B) \). But \( W \cap U_0 \) is open, so the \( h \) must coincide with \( g \) on \( W \cap U_0 \), and hence \( h = i_W(g) \). Thus \( i_W(f_s) \to i_W(g) \) and \( i_W \cup U_0(f_s) \to i_W \cup U_0(g) \), so \( U_0 \) is not maximal, a contradiction. Hence \( U_0 = M \) and \( f_i \to g \) in \( A(M, N) \). Q.E.D.

Proposition 1 is not true for general complex manifolds. Let \( M = N = \mathbb{C} \), the complex plane, and \( U = B^1 \), the open unit disk. For \( n \) a positive integer, define \( f_n(z) = (1 - 1/n)z + (1/n)z^n \). \( f_n \) approaches the identity uniformly on compact subsets of \( B^1 \), but \( f_n(z) \to \infty \) for \( |z| > 1 \).

**Corollary 2.** Let \((M, d)\) be a tight manifold and \( U \subset M \) be open. Suppose \( f: M \to M \) is holomorphic and for some subsequence, \( \{i(s)\} \) of \( Z^+ \), \( i_W(f_{i(s)}) \to \text{id}_U \). Then \( f \) is an automorphism of \( M \).

**Proof.** By Proposition 1, \( f_{i(s)} \to \text{id} \) on \( M \). This gives the conclusion by repeating verbatim the argument at the end of the proof of Theorem C [4, p. 208]. Q.E.D.

**Proposition 3.** Let \( M \) be a tight manifold with respect to some distance \( d \), \( p \in M \), and \( f: M \to M \) holomorphic with \( f(p) = p \). Then
\begin{enumerate}
  \item \( |\det df_p| \leq 1 \),
  \item \( df_p \) is the identity matrix iff \( f \) is the identity on \( M \),
  \item \( |\det df_p| = 1 \) iff \( f \) is an automorphism of \( M \).
\end{enumerate}

**Proof.** Let \( W \) be a taut [4, p. 199] neighborhood of \( p \) which is contained in some coordinate neighborhood of \( p \). By equicontinuity there is a neighborhood, \( U \), of \( p \) in \( M \) such that \( g(U) \subset W \) for any holomorphic mapping \( g: M \to M \). We may suppose in fact that \( U \) is an open ball in the coordinates about \( p \). Now (i), (ii) and the \( \Leftarrow \) part of (iii) are proved exactly as in the proof of Theorem C [4, pp. 205, 206].

For the remainder of (iii) it follows as on p. 207 of [4] that there
is a subsequence, \( \{ i(s) \} \) of \( Z^+ \) such that \( i_U(f^{i(s)}) \to \text{id}_U \). Now Corollary 2 shows \( f \) is an automorphism of \( M \). Q.E.D.

**Proposition 4.** Let \( M \) be a tight complex manifold, \( U \) be open and relatively compact in \( M \), \( f: M \to M \) holomorphic, and \( i_U(f) \) an automorphism of \( U \). Then \( f \) is an automorphism of \( M \).

**Proof.** \( M \) is tight iff \( M \) is hyperbolic, in the sense of Kobayashi [3, p. 465]. This is shown in [1, Part II: 3.8]. For \( r > 0 \), let
\[
U_r = \{ x \in U : \kappa(x, \partial U) > r \}.
\]
(Here \( \kappa \) is the Kobayashi distance on \( M \).) If \( x \in U_r \), \( x = f(y) \) for some \( y \in U \), and \( \kappa(y, U) \geq \kappa(x, f(\partial U)) \geq \kappa(x, \partial U) > r \). Thus \( U_r \subseteq f(U_r) \), and by [1, Part III: 1.5], \( f(U_r) = U_r \) and \( i_U(f) \) is an automorphism of \( U_r \).

We can choose a subsequence of \( Z^+ \), \( \{ i(m) \} \), and points \( y_m \) in \( U_r \) such that \( \kappa(f^{i(m)}(y_m), y_m) < 1/m \) for each \( m \). Since \( U_r \) is compact in \( U \) we may pass to a subsequence and assume \( y_m \to p \), where \( p \) is some point in \( \overline{U_r} \).

Now for \( x \in U_r \), \( x \) is in \( U_s \) for some \( s > 0 \), and by the argument in the first paragraph of this proof, \( f^i(x) \in U_s \) for all positive integers \( i \). Since \( U_s \) is compact in \( U \), \( \{ f^{i(m)}(x) \} \) is relatively compact in \( U \). Since \( f^{i(m)} \) is an equicontinuous family with respect to \( \kappa \), it follows from the Ascoli Theorem that there is a subsequence of \( \{ i(m) \} \), which we shall again denote by \( \{ i(m) \} \), such that \( i_U(f^{i(m)}) \to g \) in \( U \), where \( g \) is a holomorphic mapping from \( U \) to itself, and \( g(p) = p \). Since \( i_U(f) \) is an automorphism of \( U_r \) for each \( r > 0 \) and each positive integer \( i \), it follows from the relative compactness of \( U_r \) in \( U \) that \( g(U_r) = U_r \) for each \( r > 0 \), and [1, Part III: 1.5] shows that \( g \) is an automorphism of \( U_r \) for each \( r > 0 \). Hence \( g \) is an automorphism of \( U \).

By Proposition 3, \( |\det dg_p| = 1 \). From this and the argument on p. 207 of [4] it follows that for some subsequence, \( \{ k(s) \} \), of \( Z^+ gk(s) \to \text{id}_U \). It is now easily seen that for some subsequence of \( Z^+ \), \( \{ i(s) \} \), \( i_U(f^{i(s)}) \to \text{id}_U \), and Corollary 2 shows that \( f \) is an automorphism of \( M \). Q.E.D.

**References**


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