MATCHING THEOREMS FOR COMBINATORIAL GEOMETRIES

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1. Introduction. Let $G(S)$ and $G(T)$ be combinatorial geometries of finite rank on sets $S$ and $T$, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $G(S)$ and $G(T)$. By a matching from $G(S)$ into $G(T)$, we understand a one-one function $f$ from an independent set $A \subseteq S$ onto an independent set $B \subseteq T$ with $(a, f(a)) \in R$ for all $a \in A$. In this note, we present a characterization of matchings of maximum cardinality, a max-min theorem, and a number of related results. In the case when $G(S)$ and $G(T)$ are both free geometries, Theorems 1 and 2 reduce to “the Hungarian method” as introduced by Egerváry and Kuhn [1], and to the König-Egerváry theorem, respectively. Corollary 2 for the case when $G(S)$ is a free geometry and $G(T)$ arbitrary was first discovered by Rado [6] (see also Crapo-Rota [2]). When both $G(S)$ and $G(T)$ are free geometries, Corollary 2 reduces to the well-known SDR theorem.

2. Terminology. For an arbitrary geometry $G(S)$, the closure operator will be denoted by $J$ and the rank function by $r$. $(G(S), G(T), R)$ shall denote the system of the two geometries together with $R$, and $R(S') = \{ y \mid \text{there is some } x \in S' \text{ with } (x, y) \in R \}$ for $S' \subseteq S$. Let $(A, B, f)$ denote a matching from $A$ onto $B$. $M = \{ (a, f(a)) \mid a \in A \}$ is called the edge set of the matching $(A, B, f)$, and we adopt the convention $M = (A, B, f)$. The common cardinality of $A, B, M$ is called the size $v(M)$ of the matching. A support of $(G(S), G(T), R)$ is a pair $(C, D)$ of closed sets, where $C \subseteq S, D \subseteq T$, such that $(c, d) \in R$ implies at least one of $c \in C, d \in D$ holds. The order $\lambda$ of a support $(C, D)$ is defined as $\lambda(C, D) = r(C) + r(D)$. Finally, an augmenting chain with

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respect to the matching \( M = (A, B, f) \) is a sequence of \( 2n+1 \) distinct pairs \((a'_i, b'_i), (b_1, a_1), (a'_1, b'_1), \ldots, (b_n, a_n), (a'_n, b'_{n+1})\) such that

1. \( (a_i, b_i) \in M, \quad (a'_i, b'_{i+1}) \in R - M, \)
2. \( a'_i \in S - J(A), \quad b_{n+1} \in T - J(B), \)
3. \( a'_i \in J(A), \quad a'_i \notin J \left( A - \bigcup_{j=1}^{i} a_j \cup \bigcup_{j=1}^{i-1} a'_j \right), \)
4. \( b'_i \in J(B), \quad b'_i \notin J \left( B - \bigcup_{j=1}^{i} b_j \cup \bigcup_{j=1}^{i-1} b'_j \right) \)

for \( 1 \leq i \leq n. \)

3. The main results.

**Theorem 1.** A matching \( M = (A, B, f) \) in \( (G(S), G(T), R) \) is of maximum size if and only if there does not exist an augmenting chain with respect to \( M. \)

**Theorem 2.** \( \max_{M} \text{matching } \nu(M) = \min_{(C, D)} \text{support } \lambda(C, D). \)

**Brief Outline of Proof of Theorems 1 and 2.** First, it is easily seen that by means of an augmenting chain we can increase a given matching \( M, \) since by conditions (2) and (3) the sets

\[
A' = \left( A - \bigcup_{j=1}^{n} a_j \right) \cup \bigcup_{j=0}^{n} a'_j,
\]
\[
B' = \left( B - \bigcup_{j=1}^{n} b_j \right) \cup \bigcup_{j=1}^{n+1} b'_j
\]

are independent. Further, we clearly have \( \nu(M) \leq \lambda(C, D) \) for any matching \( M \) and any support \( (C, D). \)

Assume now there is no augmenting chain with respect to \( (A, B, f). \) Put \( C_0 = S - J(A), \) then \( R(C_0) \subseteq J(B). \) Let \( B_1 \) be the minimal subset of \( B \) such that \( R(C_0) \subseteq J(B_1), A_1 = f^{-1}(B_1) \) and \( C_1 = S - J(A - A_1). \) In general, having constructed \( C_{i-1}, \) we define \( B_i \) as the minimal subset of \( B \) such that \( R(C_{i-1}) \cap J(B) \subseteq J(B_i), \) and set \( A_i = f^{-1}(B_i) \) and \( C_i = S - J(A - A_i). \) This way we construct three monotonically increasing sequences of sets \( A_i, B_i, C_i \) and since all the \( B_i \)'s are contained in \( B, \) these sequences must terminate after a finite number of, say, \( m \) steps.

The crucial part of the argument consists in showing that \( R(C_n) \subseteq J(B) \) for all \( n = 0, \ldots, m. \) This is accomplished by disproving the opposite through construction of an augmenting chain with respect
to \( M \). Now since \( R(C_m) \subseteq J(B_m) \), i.e., \( R(S - J(A - A_m)) \subseteq J(B_m) \), we infer that \( J(A - A_m) \), \( J(B_m) \) constitutes a support with order equal to the size of \( M \). Thus \( M \) is a matching of maximum cardinality and the equality in Theorem 2 holds.

**Corollary 1.** For \( A \subseteq S \), define the deficiency of \( A \) as \( \delta_S(A) = r(S) - r(S - A) - r(R(A)) \), and let \( \delta_S = \max_{A \subseteq S} \delta_S(A) \). Then

\[
\max_{\text{matching } M} \nu(M) = \min_{(C, D) \text{ support}} \lambda(C, D) = r(S) - \delta_S.
\]

We have

\[
r(S) - \delta_S = r(S) - \max_{A \subseteq S} (r(S) - r(S - A) - r(R(A)))
\]

\[
= \min_{A \subseteq S} (r(S - A) + r(R(A))) = \min_{A \subseteq S} (r(A) + r(R(S - A))),
\]

and the minimum is clearly obtained by some closed set \( A \). But then \( (A, J(R(S - A))) \) is a support for \( (G(S), G(T), R) \) and the conclusion follows.

**Corollary 2 (Generalized Marriage Theorem).** Given \( (G(S), G(T), R) \), then \( \max_{\text{matching } M} \nu(M) = r(S) \) if and only if \( r(S) - r(S - A) \leq r(R(A)) \) for all \( A \subseteq S \).

**Corollary 3.** Let \( (A, B, f) \) be a matching in \( (G(S), G(T), R) \) and suppose it is not of maximum size, then there exists a matching \( (A', \cup a, B', \cup b, f') \) such that \( J(A') = J(A), J(B') = J(B) \), and \( a \in J(A'), b \in J(B') \).

This follows immediately from the definition of augmenting chains, part (3).

**Corollary 4 (See also [2], [3], [4]).** Given \( (G(S), G(T), R) \), where \( G(S) \) is a free geometry. Define a new independence structure on \( S \) by calling \( A \subseteq S \) independent if and only if there exists a matching \( (A, B, f) \) for some \( B \) and \( f \). This defines a pregeometry on \( S \), called the transversal pregeometry with respect to \( (G(S), G(T), R) \).

Corollary 3 applied to \( (G(S'), G(T), R \cap (S' \times T)) \) for \( S' \subseteq S \) shows that every independent subset \( A \subseteq S' \) as defined above can be embedded in one of maximum (and by Corollary 1, constant) size.

It should be remarked that Corollary 4 ceases to be true for arbitrary geometries \( G(S) \). The function \( r^* \) given by the definition of independent sets in Corollary 4 and by the formula in Corollary 1 as \( r^*(S') = r(S') - \delta_S \) for \( S' \subseteq S \) is unit-increasing, but fails to be semi-modular in general. For the same reason one cannot prove Theorem
2 along the lines suggested by Ore [5] although this approach works when $G(S)$ is a free geometry.

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References


