HOMOLOGICAL PROPERTIES OF THE RING OF DIFFERENTIAL POLYNOMIALS

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The ring of differential polynomials over a universal differential field (Kolchin [7]), and the ring of twisted polynomials \( \overline{F}_2[t, \rho] \), where \( \overline{F}_2 \) is an algebraic closure of \( \mathbb{Z}/2\mathbb{Z} \) and \( \rho \) is the automorphism of \( \overline{F}_2 \) defined by: \( x \rightarrow x^2 \), "localized" at the multiplicative subset \( \{ t^k \mid k \text{ an integer} \geq 0 \} \), provide examples of a principal right and left ideal domain \( R \), not a field, that is a right \( V \)-ring (i.e., each simple right \( R \)-module is injective). Such a ring was conjectured to exist by Carl Faith. Both examples are shown to have a unique simple right \( R \)-module. If \( R \) is either example, then by definition of a right \( V \)-ring, every right \( R \)-module has a maximal submodule. Bass proved that if a ring \( A \) satisfies the d.c.c. on principal left ideals, then \( A \) has a bounded number of orthogonal idempotents and every right \( A \)-module has a maximal submodule. The above examples show that the converse is false, thus answering a question raised by Bass [1, p. 470].

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1. Differential polynomials and right \( V \)-rings. Throughout this paper each ring \( R \) will be a ring with an identity element 1, and each right \( R \)-module \( M \) will be unitary in the sense that \( x1 = x \) for all \( x \in M \). Mod-\( R \) will denote the category of all right \( R \)-modules.

Definition 1. A ring \( R \) is a right \( V \)-ring (after Villamayor) in case the following equivalent conditions are satisfied:

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(1) Each simple right $R$-module is injective.
(2) Each right ideal is the intersection of maximal right ideals.
(3) $\text{Rad } M = 0$ for all $M \in \text{Mod-}R$.

1.1. PROPOSITION (FAITH). If $R$ is a prime, right noetherian, right $V$-ring, then $R$ is simple.

For a proof of Proposition 1.1 and its generalizations see Faith [3, p. 130]. Further results on the structure of right $V$-rings may also be found in Faith [4].

Let $k$ be a field with derivation $D$ and let $k[y, D]$ denote the ring of differential polynomials in the indeterminate $y$ with coefficients in $k$, i.e., the additive group of $k[y, D]$ is the additive group of the ring of polynomials in the indeterminate $y$ with coefficients in $k$, and multiplication in $k[y, D]$ is defined by: $ya = ay + D(a)$ for all $a \in k$, and its consequences. Let $f = \sum_{i=1}^{n} a_i y^i \in k[y, D]$, $a_n \neq 0$. Define the degree of $f$, $\delta(f) = n$. The following properties are immediate:

1. $\delta(fg) = \delta(f) + \delta(g)$, for all $f, g \in k[y, D]$.
2. For $f, g \in k[y, D]$, there exist $h, r \in k[y, D]$ such that $f = gh + r$ where $r = 0$ or $\delta(r) < \delta(g)$ (a similar algorithm holds on the left).

Thus, by (2), $k[y, D]$ is a principal right and left ideal domain.

Let $k$ be a field of characteristic 0 and $D$ be a derivation of $k$. A result due to Kolchin asserts the existence of a field $U \supseteq k$ and a derivation $\overline{D}$ of $U$ extending $D$ such that the equation

$$\rho(x, \overline{D}(x), \cdots, \overline{D}^{(n)}(x)) = 0, \quad n \text{ arbitrary},$$

has a solution $\xi \in U$ for all $\rho(X) \in U[X_1, \cdots, X_{n+1}] - U$. Furthermore, every homogeneous linear differential equation in $\overline{D}$ over $U$ has a nontrivial solution in $U$. Such a field $U$ is called a universal extension of $k$ or a universal differential field (Kolchin [7]).

Let $k$ be a universal differential field with derivation $D$. For the remainder of this section, we shall always denote $k[y, D]$ by $R$.

1.2. LEMMA. Given $f = \sum_{i=1}^{n} a_i y^i \in R$, $a_n = 1$, there exist $\alpha_i \in k$, $1 \leq i \leq n$, such that $f = \prod_{i=1}^{n} (y - \alpha_i)$.

PROOF. By induction on $\delta(f)$. We shall determine $\alpha, b_i \in k$, $2 \leq i \leq n$, such that

$$f = (y^{n-1} + b_2 y^{n-2} + \cdots + b_n)(y - \alpha).$$

By expanding equation (1), equating coefficients, and eliminating the $b_i$, an equation of the form

$$\rho(x, D(x), \cdots, D^{(n)}(x)) = 0$$
results. By hypothesis, there exists an $\alpha \in k$ satisfying (2).

1.2 implies, in particular, that the only irreducible elements of $R$ are those of degree 1. Hence, $V_\alpha = R/(y-\alpha)R$ is a simple right $R$-module for all $\alpha \in k$ and conversely.

1.3. Lemma. $V_\alpha = R/(y-\alpha)R$ is a divisible right $R$-module for all $\alpha \in k$.

Proof. By 1.2, it suffices to show that 

$$V_\alpha(y + \beta) = V_\alpha \quad \text{for all} \quad \alpha, \beta \in k.$$ 

Equivalently, given $h \in R$, $\delta(h) = 0$, there exist $f, g \in R$ such that

$$(1) \quad f(y + \beta) + (y + \alpha)g = h.$$ 

We shall determine $a, b \in k$ such that

$$(2) \quad a(y + \beta) + (y + \alpha)b = h.$$ 

Equation (2) is equivalent to an equation of the form

$$(3) \quad D(b) + (\alpha - \beta)b = h.$$ 

By hypothesis, there exists a $b \in k$ satisfying (3).

1.4 Theorem. The ring $R$ has the following properties:

(a) $R$ is a principal right and left ideal domain.
(b) $R$ is simple.
(c) $R$ is a right $V$-ring.
(d) $R$ is not a field.
(e) $R$ has, up to isomorphism, a unique simple right $R$-module.

Proof. (a) is obvious by properties (1) and (2) of the ring $k[y, D]$.

(b) is implied by 1.1 since $R$ is a domain.

(c) 1.3, together with the fact that divisibility is equivalent to injectivity in a principal right ideal domain (Faith [4] or Cartan-Eilenberg [2, p. 134]), implies that each simple right $R$-module is injective.

(d) Obvious.

(e) To show that $R/(y-\alpha)R \cong R/(y-\beta)R$ where $\alpha, \beta \in k$, we observe (see the proof of 1.3) that there exist nonzero $a, b \in k$ such that $a(y-\alpha) = (y-\beta)b$. The map

$$R/(y-\alpha)R \to R/(y-\beta)R$$ 

defined by

$$r + (y-\alpha)R \mapsto ar + (y-\beta)R$$

is the desired isomorphism.
2. Twisted polynomials. Let $\overline{F}_2$ denote an algebraic closure of $\mathbb{Z}/2\mathbb{Z}$ and $\rho$, the automorphism of $\overline{F}_2$ defined by: $x \mapsto x^2$. $\overline{F}_2[t, \rho]$ will denote the ring of twisted polynomials in $t$ over $\overline{F}_2$, i.e., the additive group of twisted polynomials of the indeterminate $t$ with coefficients in $\overline{F}_2$, and multiplication in $\overline{F}_2[t, \rho]$ is defined by: $ta = \rho(a)t$ for all $a \in \overline{F}_2$, and its consequences.

It is well known that $\overline{F}_2[t, \rho]$ is a principal right and left ideal domain. Furthermore, it is easy to show that the only two-sided ideals of $\overline{F}_2[t, \rho]$ are those of the form $i\overline{F}_2[t, \rho]$, $k$ an integer $\geq 0$ (Jacobson [5]).

Let $R = \overline{F}_2[t, \rho]$, $M = \{t^k | k \text{ an integer } \geq 0\}$, and $R_M = \{a/m | a \in R, m \in M\}$. For $a/t^k$, $b/t^{k+i} \in R_M$, $a/t^k = b/t^{k+i}$ if and only if $b = t^i a$. We define addition and multiplication in $R_M$ as follows: $a/t^k + b/t^k = (a + b)/t^k$ and $a/t^k b/t^l = \rho^i(a)b/t^l \rho^j$ where $\rho^i(a)$ is that element of $R$ obtained by applying $\rho^j$ to all the coefficients of $a$ (Jacobson [6, p. 211]). Clearly, $R_M$ is a simple integral domain, not a field. Moreover, $R_M$ is a principal right and left ideal domain.

2.1. Lemma. Given $r = \sum_{i=1}^{n} a_i t^i \in R$, $a_n = 1$, there exist $\alpha_i \in \overline{F}_2$, $1 \leq i \leq n$, such that $r = \prod_{i=1}^{n} (t - \alpha_i)$. In particular, the irreducible elements of $R$ are those of the form $t - \alpha$, $\alpha \in \overline{F}_2$.

Proof. Analogous to 1.2.

One readily sees that the simple right $R_M$-modules are of the form $R_M/pR_M$ where $p = t - \alpha$, $\alpha \neq 0 \in \overline{F}_2$.

2.2. Lemma. $R_M/(t - \alpha)R_M$ is a divisible right $R_M$-module for all $\alpha \in \overline{F}_2 - \{0\}$.

Proof. Analogous to 1.3.

2.3. Theorem. The ring $R_M$ has the following properties:

(a) $R_M$ is a principal right and left ideal domain.
(b) $R_M$ is simple.
(c) $R_M$ is a right $\mathcal{V}$-ring.
(d) $R_M$ is not a field.
(e) $R_M$ has, up to isomorphism, a unique simple right $R_M$-module.

Proof. Analogous to 1.4.

References


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