ORDER ALGEBRAS

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A partially ordered set \( P \) in which every pair of elements has a greatest lower bound is a semigroup, with \( pq = p \land q \), and hence is naturally associated with a semigroup algebra \( \mathbb{Z}[P] \) over the integers. For finite \( P \) Solomon has given \([3]\) a marvelously ingenious construction of an analogous sort of algebra even when \( P \) is not a semilattice and so cannot be made into a semigroup. Semigroup algebras and Solomon's "Möbius algebras" have applications in combinatorial problems involving the underlying orders.

Now in a recent study \([2]\) of valuations and Euler characteristics on lattices Rota introduced an ostensibly quite different sort of algebra he called a "valuation ring" which, rather surprisingly, plays a role like that of a semigroup algebra. More surprising, in view of their entirely different genesis and description, is that Rota's valuation ring can be shown to include Solomon's Möbius algebra as a special case.

Rota's construction, when used to associate such an algebra to a partial order \( P \) (which is only one outgrowth of his inquiry), leads in stages through several different structures. The results implicitly provide a recursive procedure for computing products in the valuation ring \( V(P) \), but give no direct formula. Solomon, on the other hand, defined his Möbius algebra by giving an explicit, if rather complicated, formula to express products of elements of \( P \) as linear combinations of \( P \)-elements. The purpose of this note is to determine from Rota's construction an explicit formula for products in \( V(P) \) which depends only on the order structure of \( P \). This will show at once that Rota's construction includes Solomon's, and it can be recast in a particularly simple form that clarifies further consequences and applications.

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1. The Rota construction. Let $L = \{S, T, \cdots\}$ be any distributive lattice under $\cup$ and $\cap$, made into a semigroup by setting $ST = S \cap T$. In the semigroup algebra $K[L]$ over a commutative ring $K$ the submodule $Q$ generated by all $S + T - ST - S \cup T$ with $S$ and $T$ in $L$ is an ideal. Since valuations on $L$ are just those functionals which are identically zero on $Q$, Rota calls the quotient $K[L]/Q$ the valuation ring $V(L, K)$. The special case of interest in this note has $L$ the lattice of "order ideals" of a partial order $(P, \preceq)$ and $K = \mathbb{Z}$, the ring of integers.

Let $P$ be such that every cone $C_p = \{q \in P : q \preceq p\}$ is finite and define $L$ to be the ring of sets generated by all cones, with $\emptyset$ added. Then $L$ is a distributive lattice whose finite elements admit the convenient height function, $ht S = |S|$ (number of elements in $S$). The quotient $\mathbb{Z}[L]/Q$ may in this case, because of its likeness to the semigroup algebra of a semigroup, be called the order algebra, $V(P)$, of $P$.

Identifying elements of $L$ with their images in $V$, Rota extends the identity defining $Q$ to give a general inclusion-exclusion formula that expresses any finite union of lattice elements as a linear combination:

\begin{equation}
S_1 \cup \cdots \cup S_r = \sum_{i=1}^{r} S_i - \sum_{i<j} S_i S_j + \sum_{i<j<k} S_i S_j S_k - \cdots.
\end{equation}

**Lemma 1.** Any $S$ of finite height in $L$ is a well-defined linear combination, $S = \sum_{p \in P} \phi_S(p) C_p$, of cones contained in $S$: that is $\phi_S(p) = 0$ unless $C_p \subseteq S$.

The proof is by induction on the height of $S$. If $ht S = 0$ then $S = \emptyset$ and this is $T + T - TT - T \cup T = 0$.

Any other element of finite height in $L$ is either a cone or a finite union of join irreducibles (i.e. cones) of finite height. Now assume the lemma for elements of height $< h$ and suppose $ht S = h$. If $S$ is not itself a cone it must be an irredundant union, $S = C_{p_1} \cup \cdots \cup C_{p_r}$, of the maximal cones contained in $S$. By $(*)$, $S = \sum C_{p_1} - \sum C_{p_i} C_{p_j} + \cdots$, where each term $C_{p_1} \cdots C_{p_k}$ on the right has height $< h$ and hence, by induction, is a well-defined linear combination of cones $C_q$ contained in it, and thus in each $C_{p_i}$. Then $S$ is the well-defined linear combination gotten by adding all such terms, and furthermore each $C_q \subseteq C_{p_i} \subseteq S$.

(If $S$ is written as a nonirredundant union of cones, which can only be done by using all the maximal $C_{p_i}$ in $S$ and other cones $C_r$ contained within some of them, it is easy to show that the added contribution from the $C_r$'s amounts to zero.)

Thus $V$ essentially consists of all linear combinations of cones and
its multiplication can be taken to define a (commutative \( Z \)-algebra) product among \( P \)-elements, say \( \circ \), by the rule: \( x \circ y = \sum_{p \in P} \phi_{xy}(p) \cdot p \) if and only if \( C_x C_y = \sum_{p \in P} \phi_{xy}(p) C_p \). This way of writing the \( V(P) \) product brings out the analogy with semigroup algebras; of course, \( V(P) \) is the integral semigroup algebra on \( P \) if and only if \( P \) is a semilattice.

2. Explicit formula for the product. Rota’s procedures show how to compute such products by working upward from minimal elements, but provide no direct way to determine \( C_x C_y \). With only these recursive techniques to build on it is natural to seek an explicit formula by repeated use of induction in the identity (*)

Suppose now that the maximal cones in a given \( C_x C_y = C_x \cap C_y \) are \( C_{p_1}, \ldots, C_{p_r} \). Then the expansion (*) can be rewritten as

\[
C_x C_y = C_{p_1} \cup \cdots \cup C_{p_r} = \sum_{i=1}^{r} C_{p_i} - \sum_{i<j} C_{p_i} C_{p_j} + \sum_{i<j<k} C_{p_i} C_{p_j} C_{p_k} - \cdots.
\]

Determining any coefficient \( \phi_{xy}(p) \) calls for further expanding each term on the right that is not already a cone until ultimately every term is reduced to a linear combination of cones, and then adding over all terms.

In fact, however, it is simpler to determine first the sum of all \( \phi_{xy}(q) \) for \( q \) in the filter \( F_p = \{ q \in P : q \geq p \} \) above \( p \). Suppose \( C_{q_1} \cdots C_{q_t} \) is any term sooner or later arising in the expansion of (**), and that its expression as a linear combination of cones is \( \sum \pi_r C_r \). Then the sum of all those \( \pi_r \) for which \( r \in F_p \) can be described as the “contribution” of the term \( C_{q_1} \cdots C_{q_t} \) to the sum \( \sigma_{xy}(p) = \sum_{q \in F_p} \phi_{xy}(q) \).

**Lemma 2.** If \( C_{q_1}, \ldots, C_{q_t} \) are cones within \( C_x \cap C_y \) then:

(a) if there is any \( i \) with \( p \geq q_i \) the contribution of \( C_{q_1} \cdots C_{q_t} \) to \( \sigma_{xy}(p) \) is 0;

(b) if \( p \leq q_i \) for each \( i \) this contribution is 1.

**Proof.** If again \( C_{q_1} \cdots C_{q_t} = \sum \pi_r C_r \) then whenever \( \pi_r \neq 0 \) for some \( r \in F_p \) it must be that \( p \leq r \leq q_i \) for each \( i \).

The proof of (b) is by induction on \( h \), the maximum of the heights \( h(C_r, C_q) \) from \( C_p \) to \( C_{q_i} \). For \( h = 0 \) the term \( C_{q_1} \cdots C_{q_t} = C_p \) does contribute 1 to \( \sigma_{xy}(p) \). Assume the lemma true whenever the maximum of these heights is less than \( h \) and now suppose that \( q \leq p \) for
each \( i \) and max \( ht(C_p, C_q) = h \). Notice that if \( t = 1 \) the term is just \( C_{p_i} \) and hence does contribute 1 to the sum.

Now with \( t > 1 \) and all \( ht(C_p, C_q) \leq h \) any cone \( C_{r_i} \) which is maximal in \( C_{q_1} \cap \cdots \cap C_{q_t} \) must have \( ht(C_p, C_{r_i}) < h \) so that for

\[
C_{q_1} \cdots C_{q_t} = C_{r_1} \cup \cdots \cup C_{r_t} = \sum_{i=1}^{t} C_{r_i} - \sum_{i<j} C_{r_i}C_{r_j} + \cdots
\]

each term on the right contributes 1 to \( \sigma_{xy}(p) \), by induction, and hence the total contribution to the sum from \( C_{q_1} \cdots C_{q_t} \) is just

\[
\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1} \binom{t}{t} = 1.
\]

**Theorem.** For any \( x, y \) and each \( p \in C_x \cap C_y \) the sum \( \sigma_{xy}(p) = 1 \).

**Proof.** Suppose \( C_{p_1}, \cdots, C_{p_r} \) are the maximal cones in \( C_x \cup C_y \) with subscripts so chosen that the first \( s \) generators \( p_1, \cdots, p_s \) are in the filter \( F_p \) and the rest are not. The terms \( C_{p_1} \cdots C_{p_l} \) of the expansion (**) can be split into two classes according as all \( p_{ij} \in F_p \) or not. Then (**) gives \( C_x C_y = \sum' + \sum'' \) where each term in the former sum \( (\sum') \) has all \( p_{ij} \in F_p \) and each term in the latter has at least one \( p_{ij} \notin p \). Now Lemma 2 shows

(a) that the whole contribution to \( \sigma_{xy}(p) \) comes from the first sum \( (\sum') \) and

(b) that each term in this sum contributes 1. But \( \sum' \) is precisely the same as the expansion by (\( * \)) of \( C_{p_1} \cup \cdots \cup C_{p_s} \) and hence

\[
\sigma_{xy}(p) = \binom{s}{1} - \binom{s}{2} + \cdots + (-1)^{s-1} \binom{s}{s} = 1.
\]

A straightforward Möbius inversion using the \( \mu \)-function of \( P \) (see [1]) now yields a simple formula for \( \phi_{xy}(p) \).

**Corollary.** For each \( x, y \) and \( p \) in \( P : \phi_{xy}(p) = \sum_{q \in P} \mu(p, q) \sigma_{xy}(q) \). Hence the product, \( o \), defined by cone multiplication is given by \( x \circ y = \sum_{p \in P} (\sum_{q \in C_x \cap C_y} \mu(p, q)) \cdot p \).

The product takes this form since \( \sigma_{xy}(q) = 1 \) or 0 according as \( q \in C_x \cap C_y \) or not.

When the order on \( P = \{x_0, x_1, x_2, \cdots \} \) can be extended to that of the natural numbers its incidence algebra \( \alpha(P) \) (see [1]) can be taken to be upper triangular matrices including the Möbius function \( M \) with \( m_{ij} = \mu(x_i, x_j) \) and its inverse the zeta function \( \zeta(z) = 1 \) or 0 as \( x_i \leq x_j \) or not. Representing each \( x_i \in P \) by the column vector with
ith component 1 and all others 0 makes $V(P)$ a left $\alpha(P) = \text{module}$ consisting of finitely nonzero vectors $x = \sum_i \xi_i x_i$ and having a convolution, $\ast$, given by $(\sum_i \xi_i x_i) \ast (\sum_j \eta_j x_j) = \sum_i \sum_j \xi_i \eta_j (x_i \circ x_j)$.

**Corollary.** If $\cdot$ denotes componentwise multiplication of column vectors, then the product of $P$-elements is given by $x_i \circ x_j = M(Zx_i \cdot Zx_j)$ and hence the convolution $x \ast y = M(Zx \cdot Zy)$.

Thus the operator $Z$ defines a convolution transform $Z(x \ast y) = Zx \cdot Zy$, and this extends to order algebras the interesting concepts and applications introduced by Tainiter [4] for finite semigroups.

**References**


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