SPECTRAL DECOMPOSITION OF ERGODIC FLOWS ON $L^p$  

BY DANIEL FIFE  

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Let $M$ be a totally σ-finite measure space and $U_s$ ($s$ real) be a one parameter group of measure—preserving transformations of $M$ satisfying appropriate measurability and continuity conditions. We let $U_s: L^p(M) \rightarrow L^p(M)$ by $U_sf = fU_s$. If $p = 2$ Stone's spectral theorem for unitary operators [2] says that there is a spectral family of projections $E_\lambda : L^2(M) \rightarrow L^2(M)$ such that for $f \in L^2(M)$

$$U_sf = \int_{-\infty}^{\infty} e^{2\pi i \lambda s} dE_\lambda f$$

from which we show that if $\psi \in L^1(R)$ and $\tilde{\psi}$ is the Fourier transform of $\psi$,

$$\int_{-\infty}^{\infty} \tilde{\psi}(\lambda) dE_\lambda f = \int_{-\infty}^{\infty} \psi(s) U_s f ds.$$  

We will say that a function is normalized at its jumps if it has only jump discontinuities and the value at each jump is the average of the values from the sides. Let $\chi_r$ be the normalized characteristic function of $(-\infty, r]$. We approximate $\chi_r$ pointwise with the Fourier transforms of $L^1$ functions and use (2) to show for $f \in L^2(M)$, $D_\lambda f = E_{\lambda-0}f + E_\lambda f - f$

$$D_\lambda f = \frac{-1}{i\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{1}{s} e^{2\pi i \lambda s} U_s f ds$$

and so $E_\lambda f = f + \frac{1}{2} D_\lambda f - \frac{1}{2} D_\lambda f$.

A slight modification of a theorem in [1] shows that $D_\lambda$ is a bounded transformation on $L^p(M)$ ($1 < p < \infty$) with the bound independent of $\lambda$. This gives

**Theorem 1.** $D_\lambda$ and hence $E_\lambda$ extend from $L^p(M) \cap L^2(M)$ to $L^p(M)$ by continuity. For $f \in L^p(M)$, $\|E_\lambda f\|_p$ is bounded uniformly in $\lambda$. $E_{\lambda-0}f = E_\lambda f$. $E_\lambda E_\lambda f = E_\lambda f$ if $\lambda \leq \tau$. $\|E_\lambda f\|_p \rightarrow 0$ as $\lambda \rightarrow -\infty$. $\|E_\lambda f - f\|_p \rightarrow 0$ as $\lambda \rightarrow +\infty$.  

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The theorems from [1] also show that convergence of the symmetrically truncated integrals to the integral in (3) is dominated and pointwise a.e.

We show that $(E_{\lambda} f, g)$ is a continuous function of $\lambda$ except for a set of jumps which is countable (and does not depend on $f, g$). Thus we can form the Stieltjes integral of any absolutely continuous function with respect to $(E_{\lambda} f, g)$ over a bounded interval. In particular we can integrate $e^{2\pi i s \lambda}$ over a bounded interval. For $f$ and $g$ in $L^2(M)$

$$(4) \quad ((E_b - E_a) U f, g) = \int_a^b e^{2\pi i s \lambda} (U f, g) ds.$$ 

We assume from now on $f \in L^p(M)$, $g \in L^{p'}(M)$, $1/p + 1/p' = 1$, $1 < p < \infty$. The absolute value of the integral in (4) is no bigger than $(b-a)^{1/p} ||f||p ||g||p'$ so the integral is a continuous function of $f$ and $g$. So is the left side of (4). Hence (4) holds for $f \in L^p(M)$, $g \in L^{p'}(M)$. Letting $a \to -\infty$ and $b \to \infty$ we get (1) for $L^p(M)$ where the integral in (1) may be taken to be a weak integral.

We now define a slight generalization of the Stieltjes integral. Suppose $h$ has support in $[a, b]$ and is continuous from the right and has a limit from the left everywhere, and suppose $\Lambda_\epsilon = \{ \lambda \in [a, b] \mid |h(\lambda) - h(\lambda - 0)| > \epsilon \}$ is finite for each $\epsilon > 0$ (for example $h_{\epsilon k}) = (E_{\lambda} f, g)$). If $\alpha$ is of bounded variation on $[a, b]$ then the integral of $h$ with respect to $\alpha$ exists in the following sense: For $\epsilon > 0$ we will only consider partitions $P \supset \Lambda_\epsilon$. If $P = \{a = \xi_0 < \xi_1 < \cdots < \xi_n = b\}$ let $S_P = \sum h_{\xi_j} [\alpha(\xi_j) - \alpha(\xi_{j-1})]$ where $\xi_{j-1} < \xi_j < \xi_j$. For such partitions and for $\epsilon > 0$, there exists $l$ depending only on $\epsilon$, $h$ and $\alpha$ such that $|S_P - S_P'| < \epsilon$ whenever mesh $P < l$ and mesh $P' < l$.

We use the above integral and some lemmas to show

**Theorem 2.** Let $\theta_i$ be Fourier multipliers for $L^p(R)$ with multiplier norms $M_i$. Assume $\theta_i$ is normalized at its jumps and has bounded variation locally.

$$(5) \quad (A(\theta_i) f, g) = \int_{-\infty}^{\infty} \theta_i(\lambda) d(E_{\lambda} f, g)$$

exists as the limit of the truncated integrals and $|A(\theta_i) f, g| \leq M_i ||f||p ||g||p'$. 

$$(6) \quad (A(\theta_1) \circ A(\theta_2) f, g) = \int_{-\infty}^{\infty} \theta_1(\lambda) \theta_2(\lambda) d(E_{\lambda} f, g)$$

i.e. $A(\theta_1) \circ A(\theta_2) = A(\theta_1 \cdot \theta_2)$.

**Theorem 3.** Suppose $\theta$ is a multiplier for $L^p(R)$, and $\theta_i, M_i$ are as
in Theorem 2 and \( \theta_j \to \theta \) pointwise and there exists \( M \) such that \( |\theta_j(\lambda)| \leq M, M_j \leq M \) for all \( j, \lambda \). Then \( \langle A(\theta_j)f, g \rangle \to \langle A(\theta)f, g \rangle \).

**Theorem 4.** If \( \phi \) is zero except at \( t_1 \cdots t_n \cdots \) and \( \sum_j \phi(t_j) < \infty \) then

\[
\int_{-\infty}^{\infty} \phi(\lambda) d(E_\lambda f, g) = \sum_{j=1}^{\infty} \phi(t_j) [(E_{t_j} f, g) - (E_{t_j-\delta_0} f, g)].
\]

These theorems allow us to integrate many multipliers with respect to \( (E_\lambda f, g) \).

We now construct two complex semigroups. For \( y \neq 0 \) let

\[
(T_{x,y} f, g) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s+iy} (U_{x-y} f, g) ds - \frac{1}{2} ((E_0 - E_{-\delta_0} f, g).
\]

Temporarily let us assume \( f \in L^p(M) \cap L^q(M), g \in L^p(M) \cap L^q(M) \) and apply (2) to get

\[
(T_{x,y} f, g) = \int_{-\infty}^{\infty} \theta_{\gamma}(\lambda) d(E_\lambda f, g) - \frac{1}{2} ((E_0 - E_{-\delta_0} f, g)
\]

where

\[
\theta_{\gamma}(\lambda) = \int_{-\infty}^{\infty} e^{2\pi i \lambda s} \frac{1}{s+iy} ds.
\]

We see from (8) that \( (T_{x,y} f, g) \) is a continuous function of \( f \in L^p(M), g \in L^p(M) \) for each \( x, y, n \) (\( y \neq 0 \)). We show that the right side of (9) is continuous in \( f \) and \( g \) and has a limit as \( n \to \infty \) by showing that \( \theta_{\gamma}(\lambda) \) and \( \theta_{\gamma}(\lambda) = \lim \theta_{\gamma}(\lambda) \) satisfy the hypotheses of Theorems 2 and 3.

To see this we subtract the truncated (at 1 and \( n \)) Hilbert transform from the truncated kernels \( 1/(s+iy) \). Thus \( (T_{x,y} f, g) = \lim (T_{x,y} f, g) \) exists. We show that \( T_{x,\nu} \circ T_{x',\nu'} = T_{x+x',\nu+\nu'} \) and that \( T_{x,\nu} \) is an analytic function of \( z = x+iy \).

\( \text{Im}(E_0 - E_{-\delta_0}) \) is the set of functions \( h \) such that \( U_s h = h \) for all \( s \).

We will assume from now on \( f \in \text{Ker}(E_0 - E_{-\delta_0}) \).

There is an equation for \( T_{x,\nu} \) like the equation for \( T_{x,\nu} \) in (9). From this we show that if \( y > 0 \) and \( f \in \text{Ker}E_0, T_{x,-\delta_0} = 0 \) so

\[
T_{x,y} f = T_{x,y} f - T_{x,-\delta_0} f = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} U_{x-y} f ds.
\]

For \( y < 0 \) \( T_{x,y} f = 0 \).

Similarly for \( f \in \text{Im}E_0, (11) \) holds if \( y < 0 \) and \( T_{x,y} f = 0 \) if \( y > 0 \).
For \( y > 0 \) and \( f \in \text{Ker}E_0 \) or \( y < 0 \) and \( f \in \text{Im}E_0 \) write \( T'_{x,y}f(\xi) \) for the integral at the right in (11) evaluated at \( \xi \in M \). Since (11) holds in \( L^p(M) \) we have for each \( x, y \) \( T'_{x,y}f(\xi) = T'_{x,y}f(\xi) \) for almost all \( \xi \in M \) but the set where \( T'_{x,y}f(\xi) \neq T'_{x,y}f(\xi) \) depends on \( (x, y) \). We show that there is a set \( M_f \subset M \) such that measure \( (M - M_f) = 0 \) and \( T'_{x,y}f(\xi) \) converges absolutely for all \( \xi \in M_f \) and all \( x, y \) \( (y \neq 0) \). \( M_f \) does not depend on \( x, y \). In Theorems 5 and 6 assume \( f \in \text{Ker}(E_0 - E_{-\delta}) \).

**Theorem 5.** The maximal function \( Sf(\xi) = \text{Sup} \{ | T'_{x,y}f(\xi) | \mid (x, y) \text{ is in a cone not tangent to the line } y = 0 \} \) is of type \( (p, p) \) \( (1 < p < \infty) \).

**Theorem 6.** For \( f \in \text{Im}E_0 \), \( T'_{x,y}f \to U_{x,O}f \) as \( (x, y) \to (x_0, 0) \) nontangentially from below.

For \( f \in \text{Ker}E_0 \), \( T'_{x,y}f \to U_{x,O}f \) as \( (x, y) \to (x_0, 0) \) nontangentially from above.

For \( h \in L^p(M) \), \( T_{x,y}h \to (E_0 - E_{-\delta})h \) as \( y \to \infty \) and \( x \) remains in any bounded set. Convergence above is \( L^p \) convergence, dominated and pointwise convergence on a subset of \( M \) having full measure.

The first two pieces of Theorem 6 say that the original group is a sort of direct sum of the two analytic semigroups we constructed.

**References**


University of Minnesota, Minneapolis, Minnesota 55455