Let $T_g$ be a closed, orientable 2-manifold of genus $g$, and let $M_g$ be the mapping class group of $T_g$, that is the group of orientation-preserving homeomorphisms of $T_g \to T_g$ modulo those isotopic to the identity. The following theorem was proved by D. Mumford in [6]: If $[M_g, M_g]$ is the commutator subgroup of $M_g$, then $A_g = M_g/[M_g, M_g]$ is a finite cyclic group whose order is a divisor of 10. We give a very brief and elementary reproof of Mumford's theorem, and at the same time improve his result to show that the order of $A_g$ is 2 if $g \geq 3$.

Generators for $M_g$ are well known, and a particularly convenient set is given by W. B. R. Lickorish in [3]. Lickorish's generators are "screw maps" about closed curves on the surface $T_g$ (the definition of a screw map is the same as that in [6]), and Lickorish shows that the screw maps about the curves \{u_i, z_i, c_i; 1 \leq i \leq g, 1 \leq j \leq g - 1\} in Figure 1 generate $M_g$.

By a well-known result [5] the group $M_g$ is isomorphic to a group of automorphism classes (cosets of the subgroup of inner automorphisms in the group of all automorphisms) of the fundamental group of the 2-manifold.

**FIGURE 1**

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Choosing generators \( \{ t_i, s_i; 1 \leq i \leq g \} \) for \( \pi_1 T_g \) as illustrated in Figure 2, and denoting screw maps about the curves \( u_i, z_i \) and \( c_i \) by \( U_i, Z_i \) and \( C_i \) respectively, the automorphisms of \( \pi_1 T_g \) corresponding to Lickorish's generators of \( M_g \) are easily determined (see [1]), and are given explicitly as follows:

\[
\begin{align*}
(1) & \quad U_i: t_i \rightarrow t_is_i & 1 \leq i \leq g \\
(2) & \quad Z_i: s_i \rightarrow t_i^{-1}t_{i+1}s_i & 1 \leq i \leq g - 1 \\
(3) & \quad C_i: s_j \rightarrow t_is_js_i^{-1} & j < i \\
(4) & \quad t_j \rightarrow t_jt_i^{-1} & j < i \\
(5) & \quad s_i \rightarrow s_is_i^{-1}
\end{align*}
\]

where it is understood that every generator of \( \pi_1 T_g \) which is not listed explicitly is unaltered by the screw maps.

This representation of \( M_g \) as a group of automorphism classes provides a very simple tool for calculation in \( M_g \). If one suspects that two sequences of screw maps are equivalent in \( M_g \), one simply calculates the induced automorphisms, and determines if they agree modulo an inner automorphism. Using this procedure, the following relations can be verified to hold in \( M_g \):

\[
\begin{align*}
(4) & \quad U_iZ_iU_i = Z_iU_iZ_i & 1 \leq i \leq g \\
(5) & \quad U_{i+1}Z_iU_{i+1} = Z_iU_{i+1}Z_i & 1 \leq i \leq g - 1
\end{align*}
\]
Relations (4)–(8) above were determined by the author in [1]; relations (9) and (10) are new, to the author’s knowledge.

We now consider the abelianizing homomorphism \( \alpha: M_g \rightarrow A_g \).

Under \( \alpha \), relation (4) goes over to

\[
 \alpha(U_i)\alpha(Z_i)\alpha(U_i) = \alpha(Z_i)\alpha(U_i)\alpha(Z_i).
\]

Since all elements in \( A_g \) commute, (11) implies:

\[
 \alpha(U_i) = \alpha(Z_i).
\]

Since similar relations link the entire set of generators of \( M_g \), we obtain immediately that \( A_g \) is a cyclic group. Denoting the single generator of \( A_g \) by \( h=\alpha(U_i) \), equations (7), (8), (9) and (10) then give

\[
 h^{(2g+1)(4)} = h^{(2g+1)(2g+2)} = 1 \quad \text{for all } g,
\]

\[
 h^{10} = 1 \quad \text{if } g \geq 3,
\]

\[
 h^{28} = 1 \quad \text{if } g \geq 4.
\]

Together these imply that the order of \( h \) is a divisor of 10 if \( g=2 \), while for \( g \geq 3 \) the order of \( h \) divides 2.

It only remains to prove that the order of \( A_g \) cannot be 1. To establish this, we make use of the well-known fact that the group \( \text{Sp}(2g, Z) \) of \( 2g \)-by-\( 2g \) symplectic matrices with integral entries is a quotient group of \( M_g \) [5], and hence the commutator quotient group of \( \text{Sp}(2g, Z) \) is a quotient group of \( A_g \). The author is grateful to J. Mennicke for pointing out that it follows from known work [2] that the commutator quotient group of \( \text{Sp}(2g, Z) \) is of order 2; hence \( A_g \) is of order 2 for all \( g \geq 3 \). For \( g=2 \) it is known that \( A_g \) is cyclic of order 10.

Some geometric insight into the proof outlined above is obtained by noting that the cyclic nature of \( A_g \) is an immediate consequence of relations (4), (5), (6). For the case of the torus \( (g=1) \) these reduce to the single relation:

\[
 U_1Z_1U_1 = Z_1U_1Z_1
\]

which is classical. Now, it is easily established that this relation re-
mains valid on a torus with \( n \) points removed. Since all pairs \((U_i, Z_i), (U_i, C_i)\) and \((Z_i, U_{i+1})\) of generators of \( M_g \) can be displayed as appropriate pairs of screw maps on subsets of \( T_g \) which are homeomorphic to a punctured torus, relations (4), (5), and (6) are seen to follow directly from the corresponding relation in \( M_1 \). The order of the single generator of \( A_g \) is determined by relations (7), (8), (9), and (10). Of these, relations (7) and (8) basically express symmetries in the geometric realization of the surface \( T_g \); relations (9) and (10) are obtained from (7) and (8) specialized to the cases \( g = 2 \) and 3 respectively, and carried over to subsets of \( T_g \) which are homeomorphic to \((T_2\text{-one point})\) and \((T_3\text{-one point})\) respectively.

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